

ON ENVELOPES OF HOLOMORPHY OF DOMAINS WITH LEVI-FLAT HATS AND THE REFLECTION PRINCIPLE

ABSTRACT. As expected (or conjectured) after the recent works of Baouendi-Ebenfelt-Rothschild, a \mathcal{C}^∞ -smooth CR diffeomorphism $h: (M, p) \rightarrow (M', p')$ between two minimal real analytic hypersurfaces in \mathbb{C}^n ($n \geq 2$) should be real analytic *if and only if* (M', p') is holomorphically nondegenerate. Constructing envelopes of holomorphy of special domains, namely “with Levi-flat hats”, we provide here a complete original proof of this statement. As a byproduct of this strategy, we derive an entirely new treatment of the essentially finite case (Baouendi-Jacobowitz-Treves’ famous theorem, 1985). More generally, we establish that the reflection mapping \mathcal{R}'_h associated to such a \mathcal{C}^∞ -smooth diffeomorphism between two minimal hypersurfaces in \mathbb{C}^n ($n \geq 1$) always extends holomorphically to a neighborhood of $p \times \bar{p}'$. This gives a generalization of the Schwarz symmetry principle to higher dimensions.

§1. INTRODUCTION AND DESCRIPTION OF THE PROOF

The present paper associates the techniques of the reflection principle and the techniques of analytic discs. Extending CR reflection objects to a Levi-flat union of Segre varieties, we come down to study the envelope of holomorphy of certain domains “with Levi-flat hats” (see §2.2 below).

1.1. Main theorem. Let $h: M \rightarrow M'$ be a smooth CR mapping between two real analytic hypersurfaces in \mathbb{C}^n ($n \geq 2$), let $p \in M$ and set $p' := h(p)$. Associated to h and to M' is the so-called *reflection function* (Xiaojun Huang’s denomination; see §1.6), an interesting invariant, more general than h . Our principal result is:

Theorem 1.2. *If h is a \mathcal{C}^∞ -smooth CR-diffeomorphism and if (M, p) is minimal, then the reflection function \mathcal{R}'_h extends holomorphically to a neighborhood of $p \times \bar{p}'$. In particular, h is real analytic at p if M' is holomorphically nondegenerate.*

1.3. Features of the classical results. The earliest result of this kind was found independently by Hans Lewy [L] and by Serguei Pinchuk [P1]: if (M, p) and (M', p') are strongly pseudoconvex, then h is real analytic. The classical proof in [L] and [P1] makes use of the so-called *reflection principle* which consists roughly to solve the map h with respect to \bar{h} and the jets of \bar{h} and to apply the Schwarz symmetry principle in a foliated union of transverse holomorphic discs. Generalizing this principle, Diederich-Webster proved in 1980 that a sufficiently smooth CR diffeomorphism is analytic at $p \in M$ if M is generically Levi-nondegenerate and the morphism of jets of Segre varieties of M' is injective (see §2 in [DW] and §1.13

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below for a definition). In 1985, Derridj studied the reflection principle for proper mappings between some model classes of weakly pseudoconvex boundaries in \mathbb{C}^2 . In 1983, Han [Ha] generalized the reflection principle for CR-diffeomorphisms between what is today called (see [BER2]) *finitely nondegenerate* hypersurfaces. In 1985, an important breakthrough was achieved by Baouendi-Jacobowitz-Treves [BJT], who proved that any \mathcal{C}^∞ CR diffeomorphism $h: (M, p) \rightarrow (M', p')$ between two real analytic CR-generic manifolds in \mathbb{C}^n which extends holomorphically to a fixed wedge of edge M , is real analytic. After Baouendi-Treves [BT2] (hypersurface case; the weakly pseudoconvex case is treated in [BeFo]), Tumanov [Tu1] (general codimension), and Baouendi-Rothschild [BR3] (necessity) it was known that the automatic holomorphic extension to a wedge of the components of h holds if and only if (M, p) is minimal in the sense of Tumanov (or equivalently, of finite type in the sense of Bloom-Graham). In the late eighties, the research on the analyticity of CR mappings has been pursued by many authors intensively. In 1988, Baouendi-Rothschild and independently, Diederich-Fornaess extended this kind of reflection principle to the non diffeomorphic case, namely for a \mathcal{C}^∞ CR map h of essentially finite hypersurfaces which is locally finite to one, or locally proper. This result was generalized in [BR2] to mappings of maximal formal generic rank on formal Segre varieties (not totally degenerate) or even more generally with non identically zero formal Jacobian determinant. Following this circle of ideas, Coupet-Pinchuk-Sukhov have pointed out that almost all the above-mentioned reflection principles come down to the fact that a certain complex analytic variety \mathbb{V}_p is zero-dimensional, which intuitively signifies that h is *finitely determined by the jets of \bar{h}* , i.e. each components h_j of h satisfies a monic Weierstrass polynomial with some coefficients being analytic functions depending on a finite jet of \bar{h} (this observation appears also in [Me1]). They stated thus a general result in the hypersurface case whose extension to a higher codimensional minimal CR-generic source M was achieved recently by Damour in [Da]. In summary, this last refinement closes up what is attainable in the spirit of the so-called *polynomial identities* devised by Baouendi-Jacobowitz-Treves, yielding a quite general sufficient condition for the analyticity of h . In the arbitrary codimensional case, this general sufficient condition can be expressed as follows. Let $\bar{L}_1, \dots, \bar{L}_m$ be a basis of $T^{0,1}M$, denote $\bar{L}^\beta := \bar{L}_1^{\beta_1} \dots \bar{L}_m^{\beta_m}$ for $\beta \in \mathbb{N}^m$ and let $\rho'_{j'}(t', \bar{t}') = 0$, $1 \leq j' \leq d'$, be a collection of real analytic equations for (M', p') . Then the complex analytic variety

$$(1.4) \quad \mathbb{V}'_p := \{t' \in \mathbb{C}^n : \bar{L}^\beta [\rho'(t', \bar{h}(\bar{p}))] = 0, \forall \beta \in \mathbb{N}^m\}.$$

is always zero-dimensional at $p' \in \mathbb{V}'_p$ in [L], [P1], [DW], [Ha], [De], [BJT], [BR1], [DF], [BR2], [BR4], [BHR], [DP], [Hu], [BER1], [BER2], [CPS1], [CPS2], [Da]. Crucially, the condition $\dim_{p'} \mathbb{V}'_p = 0$ requires (M', p') to be essentially finite.

1.5. Non essentially finite hypersurfaces. However, it is known that the finest CR-regularity phenomena come down to the consideration of a class of much more general hypersurfaces which are called *holomorphically nondegenerate* and which are generically *not essentially finite*. In 1995, Baouendi-Rothschild [BR3] exhibited this condition as a *necessary and sufficient* condition for the algebraicity of a biholomorphism between two real algebraic hypersurfaces. Thanks to the nonlocality of algebraic objects, they could assume that (M', p') is essentially finite after a small shift of p' with $\dim_{p'} \mathbb{V}'_p = 0$, thus coming down to known techniques (even in fact simpler, in the generalization to the higher codimensional case, Baouendi-Ebenfelt-Rothschild came down to a direct application of the algebraic implicit function

theorem by solving algebraically h with respect to the jets of \bar{h} [BER1]). Since then however, few works have been devoted to the study of the analytic regularity of smooth CR mapping between non-essentially finite hypersurfaces in \mathbb{C}^n . It is well-known that the main technical difficulties in the subject happen to occur in \mathbb{C}^n for $n \geq 3$ and that a great deal of such obstacles can be avoided by assuming that the target hypersurface M' is algebraic (with M real analytic), see *e.g.* the works [MM2], [Mi1,2,3], [CPS1] (in case M' is algebraic, its Segre varieties are defined all over the compactification $P_{n-1}(\mathbb{C})$ of \mathbb{C}^n , which helps much). Finally, we would like to mention the papers of Meylan [Mey], Maire and Meylan [MaMe], Meylan and the author [MM1] in this concern (nevertheless, after division by a suitable holomorphic function, the situation under study in these works is again reduced to polynomial identities).

1.6. Schwarz's reflection principle in higher dimension. In late 1996, seeking a natural generalization of Schwarz's reflection principle to higher dimension, the author (see [MM2], [Me1]) discovered the interest of the so-called *reflection function* \mathcal{R}'_h associated with h (the denomination was due to Xiaojun Huang in [Hu], but our definition in [Me1] involved one more variable and the crucial observation of its biholomorphic invariance) which appears already implicitly in [BJT]. The explicit expression of this function depends on a local defining equation for M' , but its holomorphic extendability is independent of coordinates and there are canonical rules of transformation between two reflection functions. As the author has pointed out, in the diffeomorphic case, *this function should extend without assuming any nondegeneracy condition on M' , as in the case $n = 1$* , provided M is at least of finite type in the sense of Kohn. It is easy to convince oneself that the reflection function is the right invariant to study. It has been already studied thoroughly in the algebraic and in the formal CR-regularity problems, see [Me1,3,4], [Mi2,3,4]. For instance, the formal reflection mapping associated with a formal CR equivalence between two real analytic CR-generic manifolds in \mathbb{C}^n which are minimal in the sense of Tumanov is convergent (see [Mi2,3] for partial results and [Me4] for the complete statement). If h is a holomorphic equivalence between two real algebraic CR-generic manifolds in \mathbb{C}^n which are minimal at a Zariski-generic point, then the reflection mapping \mathcal{R}'_h is algebraic (see [Mi2] for the hypersurface case and [Me3] for general codimension). So we expect that similar statements hold for smooth mappings between CR manifolds.

1.7. Statement of the results. For our part, we concentrate in this paper on smooth CR mappings between *hypersurfaces*. Thus, let as above $h: M \rightarrow M'$ be a C^∞ -smooth CR mapping between two small connected pieces of real analytic hypersurfaces in \mathbb{C}^n with $n \geq 2$. Reserving generalizations and refinements to further investigation, we shall assume here for simplicity that h is a CR-diffeomorphism. The associated reflection function \mathcal{R}'_h is defined over $M \times \bar{M}'$ as follows. Localizing M and M' at points $p \in M$ and $p' \in M'$ with $p' = h(p)$, we choose a complex analytic defining equation for M' in the form $\bar{z}' = \Theta'(\bar{w}', t')$, where $t' = (w', z') \in \mathbb{C}^{n-1} \times \mathbb{C}$ are holomorphic coordinates vanishing at p' and where the power series $\Theta'(\bar{w}', t') := \sum_{\beta \in \mathbb{N}^{n-1}} (\bar{w}')^\beta \Theta'_\beta(t')$ vanishes at the origin and converges normally in a small polydisc $\Delta_{2n-1}(0, 4r')$, $r' > 0$. By definition, the *reflection function* \mathcal{R}'_h associated with h and with such a defining function is the function $(t, \bar{v}') \mapsto \bar{\mu}' - \sum_{\beta \in \mathbb{N}^{n-1}} \bar{\lambda}'^\beta \Theta'_\beta(h(t)) =: \mathcal{R}'_h(t, \bar{v}')$, where $\bar{v}' = (\bar{\lambda}', \bar{\mu}') \in \mathbb{C}^{n-1} \times \mathbb{C}$. Clearly, this function is CR and smooth with respect to the variable $t \in M$ and

holomorphic with respect to the variable \bar{v}' near $\overline{M'}$. In case M is minimal in the sense of Tumanov at a point $p \in M$, then h and \mathcal{R}'_h extend holomorphically to one side D of M at p . Our first main result is as follows.

Theorem 1.8. *If h is a \mathcal{C}^∞ -smooth CR-diffeomorphism and if (M, p) is minimal, then the reflection function \mathcal{R}'_h extends holomorphically to a neighborhood of $p \times \overline{p'}$.*

Remark. Of course (exercise), the assumption of minimality of (M, p) can be switched to (M', p') , because (M, p) and (M', p') are CR-diffeomorphic.

Clearly, the holomorphic extendability of \mathcal{R}'_h to a neighborhood of p is equivalent to the following statement: *all the functions $\Theta'_\beta(h(t)) =: \varphi'_\beta(t)$ (an infinite number) extend holomorphically to a neighborhood of p and there exist constants $C_p, r_p > 0$ such that $|t| < r_p \Rightarrow |\varphi'_\beta(t)| < C_p^{|\beta|+1}$.* In certain circumstances, e.g. when (M', p') is Levi-nondegenerate, finitely nondegenerate or essentially finite, one deduces afterwards that h itself extends holomorphically at p . In Theorem 1.13 below, we shall derive from Theorem 1.8 above an important expected *necessary and sufficient condition* for h to be holomorphic at p .

1.9. Applications. We give essentially two important applications.

- Firstly, associated with M' , there is an invariant integer $\kappa'_{M'}$, called the *holomorphic degeneracy degree* of M' , which counts the maximal number of holomorphic vector fields with holomorphic coefficients in a neighborhood of M' which are tangent to M' and linearly independent at a Zariski-generic point. We recall that M' is called *holomorphically nondegenerate* if there does not exist a nonzero holomorphic vector field with holomorphic flow, tangent to (M', p') , i.e. if $\kappa'_{M'} = 0$. Another (equivalent) definition of $\kappa'_{M'}$ is as follows. Let $j_{t'}^k \mathcal{S}'_{\tau'}$ denote the k -jet at the point t' of the complexified Segre variety $\mathcal{S}'_{\tau'} = \{(w', z') : z' = \bar{\Theta}'(w', \tau')\}$, which induces a holomorphic map defined over the extrinsic complexification \mathcal{M}' of M' as follows:

$$(1.10) \quad \varphi'_k : \mathcal{M}' \ni (t', \tau') \mapsto j_{t'}^k \mathcal{S}'_{\tau'} = (t', \{\partial_{w'}^\beta [z' - \bar{\Theta}'(w', \tau')]\}_{|\beta| \leq k}) \in \mathbb{C}^{n + \frac{(n-1+k)!}{(n-1)! k!}}.$$

We have $\dim_{\mathbb{C}} \mathcal{M}' = 2n - 1$. Let $p'^c := (p', \bar{p}') \in \mathcal{M}'$. It is clear that there exists an integer $\chi'_{M'}$ with $0 \leq \chi'_{M'} \leq n - 1$ such that the generic rank of φ'_k equals $n + \chi'_{M'}$ for all k large enough. Then $\kappa'_{M'} = n - 1 - \chi'_{M'}$, as is shown in [BER1,2] (for the algebraic case, see [Me3]). In general, M' is biholomorphic to a product $\underline{M}' \times \Delta^{\kappa'_{M'}}$ by a $\kappa'_{M'}$ -dimensional polydisc in a neighborhood of a Zariski-generic point $q' \in M'$, where $\underline{M}' \subset \mathbb{C}^{n - \kappa'_{M'}}$ is a *holomorphically nondegenerate* hypersurface. Now, granted Theorem 1.8, we observe that the graph $\Gamma r(h) = \{(t, h(t)) : t \in (M, p)\}$ of h is clearly contained in the complex analytic set:

$$(1.11) \quad \mathcal{C}'_h := \{(t, t') \in \mathbb{C}^n \times \mathbb{C}^n : \Theta'_\beta(t') = \varphi'_\beta(t), \forall \beta \in \mathbb{N}^{n-1}\}.$$

Since the generic rank of φ'_k equals $n + \chi'_{M'}$, there exists a well-defined irreducible component \mathcal{C}''_h of \mathcal{C}'_h of dimension $n + \kappa'_{M'}$ containing the graph $\Gamma r(h)$. We deduce:

Corollary 1.12. *Let $\kappa'_{M'}$ be the holomorphic degeneracy degree of M' . Then there exists a pure closed complex analytic subset \mathcal{C}''_h of a neighborhood of $M \times M'$ in \mathbb{C}^n of dimension $n + \kappa'_{M'}$ which contains the graph of h over M . In particular, h extends as a correspondence across M if $\kappa'_{M'} = 0$.*

Remark. According to Coupet-Pinchuk-Sukhov [CPS1,2], a statement equivalent to Corollary 1.12 above would be that the transcendence degree of the field extension

$\text{Frac}(\mathcal{O}(\mathcal{V}_{\mathbb{C}^n}(M))) \rightarrow \text{Frac}(\mathcal{O}(\mathcal{V}_{\mathbb{C}^n}(M)))(h_1, \dots, h_n)$ is equal to $\kappa'_{M'}$. It can also be shown that Corollary 1.12 is equivalent to Theorem 1.8 (exercise).

• Secondly, an important particular case of Corollary 1.12 is when $\kappa'_{M'} = 0$. Assuming now that h is \mathcal{C}^∞ in order to be able to apply a theorem of Malgrange ([Ma], p. 96): *A \mathcal{C}^∞ manifold is real analytic if and only if it is contained in a real analytic set of the same dimension*), we deduce the following important result:

Theorem 1.13. *Let $h: (M, p) \rightarrow (M', p')$ be a \mathcal{C}^∞ CR diffeomorphism between two minimal real analytic hypersurfaces in \mathbb{C}^n . If M' is holomorphically nondegenerate, then h is real analytic at p .*

(Of course, real analyticity of h is equivalent to its holomorphic extendability to a neighborhood of (M, p) , by Severi's theorem, generalized to higher codimension by Tomassini.)

1.14. Necessity. Since 1995-6 (see [BR3], [BHR]), it is known that Theorems 1.3 above provide an expected *necessary and sufficient condition* in order that h is analytic (provided of course that the local CR-envelope of holomorphy of M , which already contains a side D of M at p , does not contain the other side). Indeed,

Lemma 1.15. ([BHR]) *Conversely, if (M', p') is holomorphically degenerate and if there exists a smooth CR function $\varpi: M' \rightarrow \mathbb{C}$ defined in a neighborhood of $p' \in M'$ which does not extend holomorphically to a neighborhood of p' , then there exists a CR-automorphism of (M', p') fixing p' which is not real analytic.*

1.16. Organization. To be brief, §2 presents first a thorough intuitive description (in words) of our strategy for the proof of Theorem 1.8, to which the remainder of the paper is exclusively devoted (since the above applications are classical).

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§2. PRECISE DESCRIPTION OF THE PROOF

2.1. Envelopes of holomorphy and reflection principle. According to the extendability theorem of Baouendi-Treves [BT2] (generalized to the \mathcal{C}^2 -smooth case by Trépreau), the map h in Theorem 1.8 already extends holomorphically to a one-sided neighborhood D of M at p in \mathbb{C}^n . This extension is performed by using small Bishop discs attached to M and by applying the Baouendi-Treves approximation theorem [BT1]. By the way, we would like to remind the reader of the well-known and somewhat paradoxical phenomenon of *automatic holomorphic extension of CR functions on M to both sides*, which can render the above Theorem 1.8 surprisingly trivial. Indeed, let U_M denote the (open) set of points q in M such that the envelope of holomorphy of D contains a neighborhood of q in \mathbb{C}^n (as is well-known, if, for instance, the Levi form of M has one positive and one negative eigenvalue at q , then $q \in U_M$; more generally, the *local envelope of holomorphy* of M or of the one-sided neighborhood D of M at an arbitrary point $q \in M$ is always *one-sheeted*, as can be proved using the Baouendi-Treves approximation theorem). Then clearly, the n components of our CR diffeomorphism extend holomorphically to a neighborhood of U_M in \mathbb{C}^n , as does any arbitrary CR function on M . But it remains to extend h holomorphically across $M \setminus U_M$ and the techniques of the reflection principle are then unavoidable. *Here lies the paradox*: sometimes the envelope of holomorphy trivializes the problem, sometimes it does not help. Fortunately, in the study of the

smooth reflection principle, the classical techniques do not make usually any difference between the two sets U_M and $M \setminus U_M$ and these techniques provide a *uniform method* of extending h across M , no matter the reference point p belongs to U_M or to $M \setminus U_M$ (see [L], [P1], [DW], [BJT], [BR1], [BR2], [DF], [BHR], [BER1], [BER2], [CPS1], [CPS2]). Such a uniform method seems to be quite satisfactory. On the other hand, recent deep works of Pinchuk in the study of the *geometric reflection principle* show up an accurate analysis of the relative pseudo-convex(-concave) loci of M . In [P2], [DP], [Hu], [Sha], the authors achieve the propagation of holomorphic extension of a germ along the Segre varieties of M (or the Segre sets), taking into account their relative position with respect to M and its local convexity. In such reasonings, some discussions concerning envelopes of holomorphy come down naturally in the proofs (which involve many sub-cases). However, comparing these two trends of thought, it seems to remain still really paradoxical that both phenomena contribute to the reflection principle, without an appropriate understanding of the general links between these two techniques. Guided by this observation, we have devised a new two-sided technique. In this article, we shall indeed perform the proof of Theorem 1.8 by *mixing the technique of the reflection principle together with the consideration of envelopes of holomorphy*. Further, we have been guided by a deep analogy between the various reflection principles and the results on *propagation of analyticity for CR functions along CR curves*, in the spirit of the Russian school in the sixties, of Treves' school, of Trépreau, of Tumanov, of Jöricke and others: the vector fields of the complex tangent bundle $T^c M$ being the *directions* of propagation for the one-sided holomorphic extension of CR functions, and the Segre varieties giving these directions (because $T_q^c M = T_q S_{\bar{q}}$ for all $q \in M$), one can expect that they propagate as well the analyticity of CR mappings. Of course, such a propagation property is already well-known and intensively studied since the historical work of Pinchuk [P1]. However, in the classical works (e.g. in [P2], [DP]), one propagates along a single Segre variety $S_{\bar{p}}$ and perhaps afterwards along the subsequent Segre sets if necessary (see [BER1,2], [Me2,4], [Mi3,4], [Sha]). But in this article we will propagate the analytic properties along a *bundle* of Segre varieties of M , namely *along a Levi-flat union of Segre varieties* $\Sigma_\gamma := \cup_{q \in \gamma} S_{\bar{q}}$, parametrized by a smooth curve γ transversal to $T^c M$, in analogy with the propagation of analyticity of CR functions, where one uses a *bundle of attached analytic discs*, parametrized by a curve transversal to $T^c M$. Here lies the main displacement of ideas. Let us now explain our strategy in full details and describe our proof.

2.2. Description of the proof of Theorem 1.8. To begin with, it is well-known that there exists a Zariski-open set of points (q', \bar{q}') of \mathcal{M}' at which the rank of the morphism of jets of Segre varieties φ'_k is equal to its generic rank, say $n + \chi'_{M'}$, with $0 \leq \chi'_{M'} \leq n - 1$. Thus, the jet map φ'_k is locally of constant rank at (q', \bar{q}') . In our first step, we will show that \mathcal{R}'_h is real analytic at each point $q \in M$ such that φ'_k is locally of constant rank $n + \chi'_{M'}$ at $(h(q), \overline{h(q)})$. (Of course, using the CR-diffeomorphism assumption, one can verify that $\chi_M = \chi'_{M'}$, whence also $\kappa_M = \kappa'_{M'}$, but we shall not explicitly need this fact for the proof.) In fact, if (M', p') is holomorphically nondegenerate, these points q' are the *finitely nondegenerate* points of M' , in the sense of Baouendi-Ebenfelt-Rothschild [BER2]. It will appear that our proof of the first step, a reminiscence of the Lewy-Pinchuk reflection principle, appears to be in fact a mild easy generalization of it. Now, during the second (crucial) step, to which §4–7 below are devoted, we shall extend \mathcal{R}'_h across the remaining set $E'_{M'}$ where φ'_k is not locally of constant rank. This

is where we use envelopes of holomorphy. Let $E'_{\text{na}} \subset E'_{M'} \subset M'$ (“na” for “non-analytic”) denote the closed set of points $q' \in M'$ such that \mathcal{R}'_h is not analytic in a neighborhood of $h^{-1}(q')$. If $E'_{\text{na}} = \emptyset$, Theorem 1.8 is proved. We shall therefore assume that $E'_{\text{na}} \neq \emptyset$ and we shall endeavour to derive a contradiction in several steps as follows. Following [MP], we shall first show that we can choose a particular point $p'_1 \in E'_{\text{na}}$ which is nicely disposed as follows.

Lemma 2.3. (cf. [MP]) *Let $E' \subset M'$ be an arbitrary closed subset of an everywhere minimal real analytic hypersurface $M' \subset \mathbb{C}^n$, with $n \geq 2$. If E' and $M' \setminus E'$ are nonempty, then there exists a point $p'_1 \in E'$ and a \mathcal{C}^ω one-codimensional submanifold M'_1 of M' with $p'_1 \in M'_1 \subset M'$ which is generic in \mathbb{C}^n and which divides M' near p'_1 in two open parts M'^{-}_1 and M'^{+}_1 such that E' is contained in the closed side $\overline{M'^{+}_1}$ near p'_1 .*

To reach the desired contradiction, it will suffice to prove that \mathcal{R}'_h is analytic at the point $h^{-1}(p'_1)$, where $p'_1 \in E'_{\text{na}} \cap M'_1$ is such a point as in Lemma 2.3 above. To this aim, we shall pick a long embedded real analytic arc γ' contained in M'^{-}_1 transverse to the complex tangential directions of M' , with the “center” q'_1 of γ' very close to p'_1 and we shall set $E_{\text{na}} := h^{-1}(E'_{\text{na}})$, $\gamma := h^{-1}(\gamma')$, $p_1 := h^{-1}(p'_1)$, $q_1 := h^{-1}(q'_1)$, $M_1 := h^{-1}(M'_1)$, $M_1^{-} = h^{-1}(M'^{-}_1)$ and $M_1^{+} = h^{-1}(M'^{+}_1)$. To the arc γ' , we shall associate holomorphic coordinates $t' = (w', z') \in \mathbb{C}^{n-1} \times \mathbb{C}$, $z' = x' + iy'$, such that $p'_1 = 0$ and γ' is the x' -axis (in particular, some “normal” coordinates in the sense of Chern-Moser or Baouendi-Jacobowitz-Treves, called “regular” by Ebenfelt, would be appropriate) and we shall consider the reflection function $\mathcal{R}'_h(t, \bar{v}') = \bar{\mu}' - \sum_{\beta \in \mathbb{N}^{n-1}} \bar{\lambda}'^\beta \Theta'_\beta(h(t))$ in these coordinates (w', z') . The functions $\Theta'_\beta(h(t))$ will be called the *components of the reflection function* \mathcal{R}'_h . Next, we choose coordinates $t \in \mathbb{C}^n$ near (M, p_1) vanishing at p_1 . To the \mathcal{C}^∞ -smooth arc γ , we shall associate the following \mathcal{C}^∞ -smooth Levi-flat hypersurface: $\Sigma_\gamma := \bigcup_{q \in \gamma} S_{\bar{q}}$, where $S_{\bar{q}}$ denotes the Segre variety of M associated to various points $q \in M$. Let $\Delta_n(0, r) := \{t \in \mathbb{C}^n : |t| < r\}$ be the polydisc with center 0 of polyradius (r, \dots, r) , where $r > 0$. Using the tangential Cauchy-Riemann operators to differentiate the fundamental identity which reflects the assumption $h(M) \subset M'$, we shall establish the following crucial observation.

Lemma 2.4. *There exists a positive real number $r > 0$ independent of γ' such that all the components $\Theta'_\beta(h(t)) = \left[\frac{1}{\beta!} \frac{\partial^{|\beta|}}{\partial \bar{\lambda}'^\beta} \mathcal{R}'_h(t, \bar{v}') \right]_{\bar{\lambda}'=0}$ extend as CR functions of class \mathcal{C}^∞ over $\Sigma_\gamma \cap \Delta_n(0, r)$.*

We now recall that the components $\Theta'_\beta(h(t))$ are already holomorphic in D and also holomorphic in a fixed neighborhood, say Ω , of M_1^{-} in \mathbb{C}^n , by construction of M'_1 . In particular, they are holomorphic in a neighborhood $\omega(\gamma) \subset \Omega$ in \mathbb{C}^n of $\gamma \subset M_1^{-}$. Then according to the Hanges-Treves extension theorem [HaTr], we deduce that all the components $\Theta'_\beta(h(t))$ of the reflection function extend holomorphically to a neighborhood $\omega(\Sigma_\gamma)$ of Σ_γ in \mathbb{C}^n , a (very thin) neighborhood whose size depends of course on the size of ω_γ (and the size of ω_γ goes to zero without any explicit control as the center point q_1 of γ tends to $p_1 \in E_{\text{na}}$).

To achieve the final step, we shall consider the envelope of holomorphy of $D \cup \Omega \cup \omega(\Sigma_\gamma)$ (in fact, to prevent from poly-dromy phenomena, we shall instead consider a certain subdomain of $D \cup \Omega \cup \omega(\Sigma_\gamma)$, see the details in §5 below), which is a kind of round domain $D \cup \Omega$ covered by a thin Levi-flat almost horizontal “hat-domain” $\omega(\Sigma_\gamma)$ touching the “top of the head” M along the one-dimensional arc γ

(very thin contact). Our purpose will be to show that, if the arc γ' is sufficiently close to M'_1 (whence γ is also very close to M_1), then the envelope of holomorphy of $D \cup \Omega \cup \omega(\Sigma_\gamma)$ contains the point p_1 , *even if $\omega(\Sigma_\gamma)$ is arbitrarily thin*. We will therefore deduce that all the components of the reflection function extend holomorphically at p_1 , thereby deriving the desired contradiction. By exhibiting a special curved Hartogs domain, we shall in fact prove that holomorphic functions in $D \cup \Omega \cup \omega(\Sigma_\gamma)$ extend holomorphically to the lower one sided neighborhood Σ_γ^- (roughly speaking, the “same” side as $D = M^-$); we explain below why this analysis gives analyticity at p_1 , even in the (simpler) case where p_1 belongs to the other side Σ_γ^+ . Notice that, since the order of contact between Σ_γ and M is at least equal to two (because $T_q M = T_q \Sigma_\gamma$ for every point $q \in \gamma$), we cannot apply directly some version of the edge of the wedge theorem to this situation. Another possibility (which, on the contrary, works well) would be to apply repeatedly the Hanges-Treves theorem, in the disc version given in [Tu2] (*see also* [MP]) to deduce that holomorphic functions in $D \cup \Omega \cup \omega(\Sigma_\gamma)$ extend holomorphically to the lower side Σ_γ^- , just by sinking progressively Σ_γ into D . But this would require a too complicated analysis for the desired statement. Instead, by performing what seems to be the simplest strategy, we shall use some deformations (“translations”) of the following half analytic disc attached to Σ_γ along γ . We shall consider the inverse image by h of the half-disc $(\gamma')^c \cap D'$ obtained by complexifying γ' . Rounding off the corners and reparametrizing the disc, we get an analytic disc $A \in \mathcal{O}(\Delta) \cap \mathcal{C}^\infty(\overline{\Delta})$ with $A(b^+ \Delta) \subset \gamma \subset \Sigma_\gamma$, where $b^+ \Delta := b\Delta \cap \{\operatorname{Re} \zeta \geq 0\}$, $b\Delta = \{|z| = 1\}$ and $A(1) = q_1$. It is this half-attached disc that we shall “translate” along the complex tangential directions to Σ_γ as follows.

Lemma 2.5. *There exists a \mathcal{C}^∞ -smooth $(2n-2)$ -parameter family of analytic discs $A_\sigma: \Delta \rightarrow \mathbb{C}^n$, $\sigma \in \mathbb{R}^{2n-2}$, $|\sigma| < \varepsilon$, satisfying*

- (1) *The disc $A_\sigma|_{\sigma=0}$ coincides with the above disc A .*
- (2) *The discs A_σ are half-attached to Σ_γ , namely $A_\sigma(b^+ \Delta) \subset \Sigma_\gamma$.*
- (3) *The boundaries $A_\sigma(b\Delta)$ of the discs A_σ are contained in $D \cup \Omega \cup \omega(\Sigma_\gamma)$.*
- (4) *The map $(\zeta, \sigma) \mapsto A_\sigma(\zeta) \in \Sigma_\gamma$ is a \mathcal{C}^∞ -smooth diffeomorphism from a neighborhood of $(1, 0) \in b\Delta \times \mathbb{R}^{2n-2}$ onto a neighborhood of q_1 in Σ_γ .*
- (5) *As $\gamma = h^{-1}(\gamma')$ varies and as q_1 tends to p_1 , these discs depend \mathcal{C}^∞ -smoothly upon γ' and properties (1-4) are stable under perturbations of γ' .*
- (6) *If $\gamma(0) = q_1$ is sufficiently close to M_1 , and if $p_1 \in \Sigma_\gamma^-$, then the envelope of holomorphy of (an appropriate subdomain of) $D \cup \Omega \cup \omega(\Sigma_\gamma)$ contains p_1 .*

Consequently, using these properties (1-6) and applying the continuity principle to the family A_σ , we shall obtain that the envelope of holomorphy of $D \cup \Omega \cup \omega(\Sigma_\gamma)$ (in fact of a good subdomain of it, in order to assure monodromy) contains a large part of the side Σ_γ^- of Σ_γ in which $D(= M^-)$ lies. In the case where p_1 lies in this side Σ_γ^- , and provided that the center point q_1 of γ is sufficiently close to p_1 , we are done: the components of the reflection function extend holomorphically at p_1 . Of course, it can happen that p_1 lies in the side Σ_γ^+ or in Σ_γ . In fact, the following tri-chotomy is in order to treat this case. To apply Lemma 2.5 wisely, and to complete the study of our situation, we shall indeed distinguish three cases. *Case I:* the Segre variety $S_{\bar{p}_1}$ cuts M_1^- along an infinite sequence of points $(q_k)_{k \in \mathbb{N}}$ tending towards p_1 ; *Case II:* the Segre variety $S_{\bar{p}_1}$ does not intersect M_1^- in a neighborhood of p_1 and it goes under M_1^- . *Case III:* the Segre variety $S_{\bar{p}_1}$ does not intersect M_1^- in a neighborhood of p_1 and it goes over M_1^- . In the first case,

choosing the point q_1 above to be one of the points q_k which is sufficiently close to p_1 , and using the fact that p_1 belongs to $S_{\bar{q}_1}$ (because $q_1 \in S_{\bar{p}_1}$), we have in this case $p_1 \in \Sigma_\gamma$ and the holomorphic extension to a neighborhood $\omega(\Sigma_\gamma)$ already yields analyticity at p_1 (in this case, we have nevertheless to use Lemma 2.5 to insure monodromy of the extension). In the second case, we have $S_{\bar{p}_1} \cap D \neq \emptyset$. We then choose the center point q_1 of γ very close to p_1 . Because we then have a uniform control of the size of $\omega(\Sigma_\gamma)$, we again get that p_1 belongs to $\omega(\Sigma_\gamma)$ and Lemma 2.5 is again used to insure monodromy. In the third (a priori more delicate) case, by a simple calculation, we shall observe that p_1 always belong to Σ_γ^- and Lemma 2.5 applies to yield holomorphic extension and monodromy of the extension, and we are done in the three cases. In conclusion to this presentation, we would like to say that some unavoidable technicalities that we have not mentioned here will render the proof a little bit more complicated (especially about the choice of q_1 sufficiently close to p_1 , about the choice of γ and about the smooth dependence with respect to γ of Σ_γ and of A_σ).

§3. EXTENSION ACROSS A ZARISKI DENSE OPEN SUBSET OF M

The starting point of the proof is to show that the reflection function extends holomorphically across M at a Zariski-generic point of M . This is done by performing a very easy generalization of the classical Lewy-Pinchuk reflection principle. Thus, let φ'_k denote the morphism of k -jets of Segre varieties of M' expressed in some coordinate system as in (1.10) above. The generic rank of the holomorphic map φ'_k stabilizes for k large enough. We let $n + \chi'_{M'}$, $0 \leq \chi'_{M'} \leq n - 1$, denote this generic rank and we fix such a large integer k definitely. We set $\kappa' := \kappa'_{M'} = n - 1 - \chi'_{M'}$. We denote by $E'_{M'}$ the set of points $p' \in M'$ in a neighborhood of which the Segre morphism φ'_k is not of constant rank $n + \chi'_{M'}$. Using the representation (1.10) of φ'_k in coordinates, it can be checked there exists a complex analytic set \mathcal{E}' , closed in a neighborhood of M' , such that $E'_{M'} = \mathcal{E}' \cap M'$ (exercise; cf. [Me1]; we believe that \mathcal{E}' is the intrinsic complexification of $E'_{M'}$, but we have no proof of it; anyway, we need not such a result). We set $G'_{M'} := M' \setminus E'_{M'}$. The following fact is well known in the subject (see e.g. [BER2], [Me3]).

Lemma 3.1. *For each point $q' \in G'_{M'}$, there exists a local coordinate system $(w', v', z') \in \mathbb{C}^{n-1-\kappa'} \times \mathbb{C}^{\kappa'} \times \mathbb{C}$ vanishing at q' such that the defining equations of (M', q') take the simple form $\bar{z}' = \Theta'(\bar{w}', w', z')$, where the analytic function Θ' is independent of the coordinates (v', \bar{v}') .*

In other words, near such a Zariski-generic point $q' \in G'_{M'}$, then M' is biholomorphic to the product $\underline{M}' \times \Delta^{\kappa'}$, where the hypersurface $\underline{M}' \subset \mathbb{C}^{n-\kappa'}$, equipped with coordinates (w', z') , is given by the equation $\bar{z}' = \Theta'(\bar{w}', w', z')$. As $q' \in G'_{M'}$, we have furthermore that the morphism φ'_k of k -jets of Segre varieties of \underline{M}' is immersive at \underline{q}' , i.e. has rank maximal equal to $2(n - \kappa') - 1$. In other words, the hypersurface $\underline{M}' \subset \mathbb{C}^{n-\kappa'}$ is finitely nondegenerate at \underline{q}' ([BER2]). The main result of this paragraph can be summarized as follows. It then shows that \mathcal{R}'_h is analytic at each point of $G'_{M'}$, since M is minimal at q for all q in a small neighborhood of p . In the following assertion, we use \mathcal{C}^1 -smoothness only (as noticed in [BJT], [HMM], in the \mathcal{C}^1 case, one can prove more generally that the components of the reflection function extend holomorphically across p in the essentially finite case; we prove the elementary Lemma 3.2 just for completeness).

Lemma 3.2. *Let $h: (M, q) \rightarrow (M', q')$ be a \mathcal{C}^1 -smooth CR-diffeomorphism. If (M, q) is minimal and the Segre morphism of M' is of constant maximal rank at q' (i.e. $q' \in G_{M'}$), then \mathcal{R}'_h extends holomorphically at q .*

Proof. We choose coordinates (w', v', z') as above, we split the components of the mapping as $h = (g, l, f)$ accordingly and we set $\underline{h} := (g, f)$. The reflection function $\mathcal{R}'_h = \bar{\mu}' - \sum_{\beta \in \mathbb{N}^{n-1-\kappa'}_{M'}} \bar{\lambda}'^\beta \Theta'_\beta(\underline{h})$ is independent of the κ' components $l = (l_1, \dots, l_{\kappa'})$ of the mapping h and \mathcal{R}'_h depends only on the partial map \underline{h} . We shall therefore immediately deduce that \mathcal{R}'_h is analytic at 0, from the following assertion. \square

Lemma 3.3. *The components of $\underline{h} = (g, f)$ extend holomorphically at q .*

Proof. The proof is an easy generalization of the Lewy-Pinchuk reflection principle, which corresponds to the case $\kappa' = 0$ and M' being 2-nondegenerate at q' in the sense of [BER2]. Let L_1, \dots, L_{n-1} be a commuting basis of $T^{1,0}M$ with real analytic coefficients, for instance $L_j = \frac{\partial}{\partial w_j} + \bar{\Theta}_{w_j}(w, \bar{t}) \frac{\partial}{\partial \bar{z}}$, $j = 1, \dots, n-1$, where $z = \bar{\Theta}(w, \bar{t})$ is a defining equation for (M, q) with coordinates $t = (w, z)$ vanishing at q . Since h is a \mathcal{C}^1 -smooth CR diffeomorphism, after an eventual linear change of coordinates near M , we can assume that the determinant $\det(\bar{L}_j \bar{g}_k(t))_{1 \leq j, k \leq n-1-\kappa'}$ is nonzero at the point $\bar{t} = 0$. Applying the derivations $\bar{L}_1, \dots, \bar{L}_{n-1-\kappa'}$ to the fundamental identity $\bar{f} = \Theta'(\bar{g}, \underline{h})$ and using Cramer's rule as in [P1], we deduce that for each multiindex $\beta \in \mathbb{N}^{n-1-\kappa'}$ with $|\beta| = 1$, then there exists an analytic function $\underline{\Omega}_\beta$ such that, for all $t \in M$ in a neighborhood of 0, we have:

$$(3.4) \quad [\partial_{\bar{w}}^\beta \Theta'](\bar{g}(t), \underline{h}(t)) = \underline{\Omega}_\beta(t, \bar{t}, \{\partial_t^\alpha \bar{h}(t)\}_{|\alpha| \leq 1}) =: \underline{\omega}_\beta(t, \bar{t}).$$

Here, the right-hand sides of (3.4) are *a priori* only \mathcal{C}^0 -smooth with respect to $t \in M$, but we observe readily that the left-hand sides are in fact \mathcal{C}^1 . Therefore the right-hand sides $\underline{\omega}_\beta(t, \bar{t})$ are in fact \mathcal{C}^1 over M . Thus, we shall be allowed to apply again the derivations $\bar{L}_1, \dots, \bar{L}_{n-1-\kappa'}$ to the equations (3.4). To begin with, let $D = M^-$ denote the local one-sided neighborhood of M to which the components of h have a holomorphic extension, by the Baouendi-treves extension theorem [BT2] (we have assumed that (M, q) is minimal) and let M^+ denote the other side, approximatively symmetric to D . As in the Lewy-Pinchuk reflection principle, using the one-dimensional Schwarz reflection in the complex lines $\{w = ct\}$ which are transverse to M , we observe that the functions $\underline{\omega}_\beta$ extend continuously to M^+ as functions which are partially holomorphic with respect to the transverse variable z . Since their boundary value $\underline{\omega}_\beta$ is \mathcal{C}^1 on the boundary (M, q) thanks to the above observation, a known regularity principle in one-dimensional complex analysis (see [Hö], [BJT], [HMM]) shows that the partial holomorphic extension with respect to z of $\underline{\omega}_\beta$ to M^+ is in fact of class \mathcal{C}^1 over $M^+ \cup M$. We can thus re-apply the derivations \bar{L}_j 's to the identity $[\partial_{\bar{w}}^\beta \Theta'](\bar{g}(t), \underline{h}(t)) = \underline{\omega}_\beta(t, \bar{t})$ and then use again Cramer's rule, to deduce that for each $\beta \in \mathbb{N}^{n-1-\kappa'}$ with $|\beta| = 2$, there exists a \mathcal{C}^0 function $\underline{\omega}_\beta(t, \bar{t})$, extending holomorphically with respect to z in M^+ such that $[\partial_{\bar{w}}^\beta \Theta'](\bar{g}(t), \underline{h}(t)) = \underline{\omega}_\beta(t, \bar{t})$. Again, we deduce from this relation that such $\underline{\omega}_\beta$'s for $|\beta| = 2$ are in fact \mathcal{C}^1 over $M^+ \cup M$ and we get in conclusion a similar identity by induction on β for all $\beta \in \mathbb{N}^{n-1-\kappa'}$. Let us write this relation in the form:

$$(3.5) \quad \Theta'_\beta(\underline{h}(t)) + \sum_{\gamma \in \mathbb{N}_*^n} \bar{g}(\bar{t})^\gamma \Theta'_{\beta+\gamma}(\underline{h}(t)) (\beta + \gamma)! / [\beta! \gamma!] = \underline{\omega}_\beta(t, \bar{t}) / [\beta!],$$

where $\nu := n - 1 - \kappa'$ and $\mathbb{N}_*^\nu := \mathbb{N}^\nu \setminus \{0\}$. Now, it is easy to check that the map $\underline{t}' \mapsto (\Theta'_\beta(\underline{t}'))_{|\beta| \leq k}$ is immersive at 0, since the map $(\underline{t}', \underline{t}) \mapsto (\underline{t}', ([\partial_{\bar{w}'}^\beta \Theta'](\bar{w}', \underline{t}'))_{|\beta| \leq k})$ is immersive at 0 on \underline{M}' , by assumption, for k large enough. In eqs. (3.5), we consider the terms $\bar{g}(\bar{t})^\gamma$ with $\bar{g}(0) = 0$ to be \mathcal{C}^1 over $M^+ \cup M$ and partially holomorphic with respect to z , as are the left-hand side terms $\underline{\omega}_\beta(t, \bar{t})$. Applying therefore the implicit function theorem to eqs. (3.5), we deduce that there exists a \mathcal{C}^1 mapping $a = (a_1, \dots, a_{n-\kappa'})$ over M which extend partially holomorphically with respect to z into M^+ such that $\underline{h}(t) = a(t, \bar{t})$ when $t \in M$. As in the classical Lewy-Pinchuk reflection principle, this proves that \underline{h} extends holomorphically at 0. The proofs of Lemmas 3.2 and 3.3 are complete now. \square

3.6. On equivalences of hypersurfaces. As an application of Lemma 3.2, we mention here the following easy and very useful corollary (see §7 below).

Lemma 3.7. *Let $h: (M, p) \rightarrow (M', p')$ be a \mathcal{C}^k CR-diffeomorphism, $1 \leq k < \infty$. If (M, p) is minimal, if the Segre morphism of M' is of constant rank in a neighborhood of (p', \bar{p}') and if M' is given by an arbitrary equation of the form $\bar{z}' = \sum_{\beta \in \mathbb{N}^{n-1}} \bar{w}'^\beta \Theta'_\beta(t')$, then there exists a holomorphic map $H: (M, p) \rightarrow (M', p')$ whose k -th jet at p coincides with the k -jet at p of h such that $\Theta'_\beta(H(t)) \equiv \Theta'_\beta(h(t))$ for all $\beta \in \mathbb{N}^{n-1}$. Furthermore, if $\kappa'_{M'} = 0$, then H is unique, $H \equiv h$ and h is analytic. Finally, for q running in a neighborhood of p , there exists such a family of equivalences $H_q: (M, q) \rightarrow (M', q')$ depending \mathcal{C}^k -smoothly with respect to q .*

Remark. We believe that Lemma 3.7 holds true without the restriction that the Segre morphism is locally of constant rank, but we have no proof of that (on the other hand, if h is \mathcal{C}^∞ , we have a proof of it, just by considering the formal map induced by the Taylor series of h at p and by applying the “Corollaire 2.7” of [Me4]).

Proof. The biholomorphic invariance of Segre varieties entails that this property $\Theta'_\beta(H(t)) \equiv \Theta'_\beta(h(t))$ for all $\beta \in \mathbb{N}^{n-1}$ is satisfied for every system of coordinates if and only if it is satisfied for a single such system of coordinates. Now, in the coordinates given by Lemma 3.1, we have established that $h = (g, l, f)$ has the property that the components g and f are analytic. It suffices therefore to choose $H_q := (g, j_q^k l, f)$ for q in a neighborhood of p . \square

§4. LAYOUT OF A TYPICAL POINT OF NON ANALYTICITY

Thus, we already know that \mathcal{R}'_h is analytic over the open dense subset $h^{-1}(G'_{M'})$ of M . We notice that using the considerations of §2 above, it can be shown that $G_M = h^{-1}(G'_{M'})$, but we shall not need this fact. It remains to show that \mathcal{R}'_h is analytic at each point $p := h^{-1}(p')$ for $p' \in E'_{M'}$. This objective constitutes the principal task of the demonstration. In fact, we shall prove a slightly more general semi-global statement which we can summarize as follows.

Theorem 4.1. *Let $h: (M, p) \rightarrow (M', p')$ be a \mathcal{C}^∞ -smooth CR-diffeomorphism of connected everywhere minimal real analytic hypersurfaces in \mathbb{C}^n . If the local reflection mapping \mathcal{R}'_h associated to some coordinate system for (M', p') is analytic at one point q of (M, p) , then it is analytic all over (M, p) .*

Remark. If the small piece of hypersurface (M, p) is minimal at p , then shrinking it if necessary, it will be minimal at every point, because the condition $\text{Lie}_q(T^c M) = T_q M$ is an open condition. Thus, of course we assume that (M, p) is minimal at

every point and in fact, our Theorem 4.1 holds true if, instead of a germ, we consider a map of *globally minimal* (hence connected) “large” real analytic hypersurfaces.

4.2. Construction of a generic wall. In §2 above, we have already shown that \mathcal{R}'_h is analytic at each point of the nonempty open set $h^{-1}(G'_{M'})$, thus Theorem 4.1 implies our goal, Theorem 1.8. To prove Theorem 4.1, let us denote by E'_{na} the closed set of points $p' \in M'$ such that \mathcal{R}'_h is not analytic in a neighborhood of $h^{-1}(p')$. If $E'_{\text{na}} = \emptyset$, we are done. We suppose therefore by contradiction that $E'_{\text{na}} \neq \emptyset$. We shall reach a contradiction by showing that there exists *a (i.e. at least one) point* $p'_1 \in E'_{\text{na}}$ such that \mathcal{R}'_h is analytic at $h^{-1}(p'_1)$. As in Lemma 2.3, this point p'_1 will belong to a generic one-codimensional submanifold $M'_1 \subset M'$, a kind of “wall” in M' dividing M' locally into two open sides, which will be disposed conveniently in order that one open side of the “wall”, say M'^{-}_1 , will contain only points where \mathcal{R}'_h is already real analytic. To show the existence of such a point $p'_1 \in E'_{\text{na}}$ and of such a manifold (“wall”) M'_1 , we shall proceed as in [MP, Lemma 2.3] (the reader is referred to this paper for more details about this construction). As M' is minimal at every point, it is certainly globally minimal, in the sense that the CR-orbit of every point of M' is M' itself in the whole. We thus choose an arbitrary point $r' \in M' \setminus E'_{\text{na}} \neq \emptyset$ (it is nonempty, because we already know that $M' \setminus E'_{\text{na}} \supset M' \setminus E'_{M'} = G'_{M'} \neq \emptyset$). The CR-orbit of r' is equal to M' in the whole. Thus there exists a piecewise differentiable \mathcal{C}^ω curve γ' running in complex tangential directions to M' with origin r' and endpoint a point $q' \in E'_{\text{na}}$. After shortening γ' , we can assume that γ' is a smoothly embedded real segment with boundary with $q' = E'_{\text{na}} \cap \gamma'$ and that γ' extends a little bit further as the integral curve of a \mathcal{C}^ω section L' of $T^c M'$. Then using the dynamical flow of L' and a one parameter family of real $(2n-2)$ -dimensional spheroids B_s , $s \in \mathbb{R}$, which are stretched along the flow lines of L' in M' and centered at a point $r' \in \gamma' \setminus E'_{\text{na}}$ sufficiently close to q' (see [MP, Lemma 2.3]), we can choose M'_1 to be a piece of the boundary of the first spheroid B_{s_1} which happens to touch E'_{na} at a point p'_1 . Furthermore, the $(2n-2)$ -dimensional spheroid B_{s_1} is *generic* in \mathbb{C}^n at p'_1 by construction, since we can assume in this construction that the vector field L' is tangent to the spheroids B_s *only* along a fixed $(2n-3)$ -dimensional spheroid contained in $M' \setminus E'_{\text{na}}$ which is independent of s . In summary, it suffices now for our purposes to establish the following assertion.

Theorem 4.3. *Let $p'_1 \in E'_{\text{na}}$ and assume that there exists a \mathcal{C}^ω one-codimensional submanifold M'_1 with $p'_1 \in M'_1 \subset M'$ which is generic in \mathbb{C}^n such that $E'_{\text{na}} \setminus \{p'_1\}$ is completely contained in one of the two open sides of M' divided by M'_1 at p'_1 , say in M'^{+}_1 , and such that \mathcal{R}'_h is analytic at each point of the other side M'^{-}_1 . Then the reflection function \mathcal{R}'_h extends holomorphically at $h^{-1}(p'_1)$.*

Now, an elementary reasoning using only linear changes of coordinates and Taylor’s formula shows that, after an eventual change of the manifold M'_1 in a new manifold M''_1 which is bent quadratically in the side M'^{-}_1 , we can assume that M' is given by the equation $z' = \bar{\Theta}'(w', \bar{t}')$, that M'_1 is given by the two equations $z' = \bar{\Theta}'(w', \bar{t}')$, $u'_1 = -[v_1'^2 + |w'_* \bar{w}'_*|^2 + x^2]$, where $w'_1 = u'_1 + iv'_1$, $w'_* = (w'_2, \dots, w'_{n-1})$ and that the side M'^{-}_1 is given by :

$$(4.4) \quad M'^{-}_1 : \quad \{(w', z') \in M' : u'_1 < -[v_1'^2 + |w'_*|^2 + x^2]\},$$

in coordinates (w', z') centered at p'_1 . We set $p_1 := h^{-1}(p'_1)$, $M_1 := h^{-1}(M'_1)$ which is a \mathcal{C}^∞ -smooth one-codimensional generic submanifold of M . By assumption, the

reflection function \mathcal{R}'_h associated with these coordinates is already holomorphic at each point of the side of automatic extension $D = \{(w, z) \in \mathbb{C}^n : y < h(w, \bar{w}, x)\}$. and it is also real analytic at each point of $M_1^- \subset M$. Let us write this more precisely. Without loss of generality, we can assume that Θ and Θ' both converge in $\Delta_n(0, 4r)$, with $r > 0$. Let $(\Psi'_{p'})_{p' \in M'}$ denote a family of biholomorphisms sending $p' \in M'$ to 0, holomorphically parametrized by $p' \in \Delta_n(0, r)$ with $\Psi'_0 = \text{Id}$. Let $\bar{z}' = \Theta'_{p'}(\bar{w}', t')$ denote the equation of $\Psi'_{p'}(M')$. By saying that \mathcal{R}'_h (associated with $p' = 0$) extends holomorphically across M at each point of M_1^- , we mean precisely that each reflection function $\mathcal{R}_{p', h_{p'}}$ in these coordinates extends holomorphically to a neighborhood of $p \times 0$, for every point $p \in M_1^-$. Using then an explicit representation for $\Psi'_{p'}$ and achieving elementary calculations with power series, we obtain the following concrete characterization in which a single coordinate system is considered.

Lemma 4.5. *There exists a neighborhood Ω of M_1^- and a lower semi-continuous positive function $r(p) > 0$ such that for each point $p \in M_1^-$, the polydisc $\Delta_n(p, r(p))$ is contained in Ω , the components $\Theta'_\beta(h(t))$ of the reflection function associated with a fixed coordinate system converge in $\Delta_n(0, r(p))$ and they satisfy a Cauchy estimate $|\Theta'_\beta(h(t))| < C_p^{|\beta|+1}$ for $|t - p| < r(p)$.*

Now start our principal constructions. As explained in §2.2, we intend to study the envelope of holomorphy of the union of D together with an arbitrary thin neighborhood of a Levi-flat hypersurface Σ_γ . We need real arcs and analytic discs.

§5. ENVELOPES OF HOLOMORPHY OF DOMAINS WITH LEVI-FLAT HATS

5.1. A family of real analytic arcs. To begin with, we choose coordinates t and t' as above near M and near M' in which p_1 and p'_1 are the origin and in which the equations of M and of M' are given by $M : z = \bar{\Theta}(w, \bar{t})$ and $M' : z' = \bar{\Theta}'(w', \bar{t}')$. We can assume that the power series defining $\bar{\Theta}$ and $\bar{\Theta}'$ converge normally in the polydisc $\Delta_{2n-1}(0, 4r)$, for some $r > 0$. In the target space, we now define a convenient, sufficiently rich, family of embedded real analytic arcs $\gamma'_{w'_{q'}, x'_{q'}}(s')$, depending on $(2n - 1)$ very small real parameters $(w'_{q'}, x'_{q'}) \in \mathbb{C}^{n-1} \times \mathbb{R}$ satisfying $|w'_{q'}| < \varepsilon$, $|x'_{q'}| < \varepsilon$, where $\varepsilon \ll r$, with “time” s' satisfying $|s'| \leq 2r$, and which are all transverse to the complex tangential directions of M' (shrinking r if necessary), as follows :

$$(5.2) \quad \left\{ \begin{array}{l} \gamma'_{w'_{q'}, x'_{q'}} := \left\{ (u'_{1,q'} - s'^2 - (v'_{1,q'} + s')^2 - |w'_{*q'}|^2 - x'^2_{q'} + i[v'_{1,q'} + s']), \right. \\ \left. w'_{*q'}, x'_{q'} \in M' : s' \in \mathbb{R}, |s'| \leq 2r \right\}. \end{array} \right.$$

It can be straightforwardly checked that the following properties hold :

- (1) *The mapping $(w'_{q'}, x'_{q'}) \mapsto \gamma'_{w'_{q'}, x'_{q'}}(0)$ is a C^ω real diffeomorphism onto a neighborhood of 0 in M' . Furthermore, the inverse image of M'_1 and of M'^{-}_1 correspond to the sets $\{u'_{1,q'} = 0\}$ and $\{u'_{1,q'} < 0\}$, respectively.*
- (2) *For $u'_{1,q'} < 0$, we have $\gamma'_{w'_{q'}, x'_{q'}} \subset \subset M'^{-}_1$.*
- (3) *For $u'_{1,q'} = 0$, we have $\gamma'_{w'_{q'}, x'_{q'}} \cap M'_1 = \{\gamma'_{w'_{q'}, x'_{q'}}(0)\}$.*
- (4) *The order of contact of $\gamma'_{w'_{q'}, x'_{q'}}$ with M'_1 at the point $\gamma'_{w'_{q'}, x'_{q'}}(0)$ equals 2.*

5.3. Inverse images. Since h is a \mathcal{C}^∞ CR diffeomorphism, we get in M a family of \mathcal{C}^∞ arcs, $h^{-1}(\gamma'_{w'_q, x'_q})$. It is clear that we get a parameterized family of arcs depending in a \mathcal{C}^∞ fashion with respect to some variables $(w_q, x_q, s) \in \mathbb{C}^{n-1} \times \mathbb{R} \times \mathbb{R}$, which we will denote by $\gamma_{w_q, x_q}(s)$ accordingly (by the index notation \cdot_{w_q, x_q} , we mean that the arc is parametrized by its center point $\gamma_{w_q, x_q}(0) \in M$, which covers in a diffeomorphic way a neighborhood of 0 in the manifold M equipped with coordinates (w, x) : this is why we shall maintain in the sequel such a notation). Of course, after adapting a bit the domains of variation (or shrinking a bit ε and r), we can suppose that $|w_q| < \varepsilon$, $|x_q| < \varepsilon$, $|s| \leq 2r$ and again $\varepsilon \ll r$. Then the \mathcal{C}^∞ arcs γ_{w_q, x_q} satisfy the two properties (1), (2), (3) and (4) above with respect to M_1 . In particular,

- (5) *There exists a continuous function $r(\varepsilon)$ with $0 < r(\varepsilon) \leq 2r$ and tending to 0 with ε such that, for all (w_q, x_q) with $|w_q|, |x_q| < \varepsilon$, we have:*

$$(5.4) \quad \{\gamma_{w_q, x_q}(s) : r(\varepsilon) \leq s \leq 2r\} \subset\subset M_1^-.$$

This property will be of interest later, when envelopes appear on scene.

5.5. Construction of a family of Levi-flat hats. Next, if γ is a \mathcal{C}^∞ -smooth arc in M transverse to $T^c M$ at each point, we can construct the union of Segre varieties attached to the points running in γ : $\Sigma_\gamma := \bigcup_{p \in \gamma} S_{\bar{p}}$. For various arcs γ_{w_q, x_q} , we obtain various sets $\Sigma_{\gamma_{w_q, x_q}}$ which are in fact \mathcal{C}^∞ -smooth Levi-flat hypersurfaces in a neighborhood of γ_{w_q, x_q} . The uniformity of the size of such neighborhoods follows immediately from the smooth dependence with respect to (w_q, x_q) . What we shall need in the sequel can be then summarized as follows.

Lemma 5.6. *After shrinking r if necessary, there exists $\varepsilon > 0$ with $\varepsilon \ll r$ such that, if the parameters (w_q, x_q) satisfy $|w_q|, |x_q| < \varepsilon$, then the set $\Sigma_{\gamma_{w_q, x_q}}$ is a closed \mathcal{C}^∞ -smooth (and \mathcal{C}^∞ -smoothly parametrized) Levi-flat hypersurface of $\Delta_n(0, r)$.*

5.7. Two families of half-attached analytic discs. Let us now define inverse images of analytic discs. Complexifying the \mathcal{C}^ω arcs $\gamma'_{w'_q, x'_q}$, we obtain transverse holomorphic discs, closed in $\Delta_n(0, 3r/2)$, of which one half part penetrates inside $D' := h(D)$. Uniformly smoothing out the corners of such half discs, using Riemann's conformal mapping theorem and then an automorphism of Δ , we can easily construct a \mathcal{C}^ω -parameterized family of analytic discs $A'_{w'_q, x'_q} : \Delta \rightarrow \mathbb{C}^n$ which are \mathcal{C}^∞ up to the boundary $b\Delta$ such that, if we denote $b^+\Delta := b\Delta \cap \{\operatorname{Re} \zeta \geq 0\}$ (and $b^-\Delta := b\Delta \cap \{\operatorname{Re} \zeta \leq 0\}$), then we have $A'_{w'_q, x'_q}(1) = \gamma'_{w'_q, x'_q}(0)$ and also:

$$(5.8) \quad A'_{w'_q, x'_q}(b^+\Delta) \subset \gamma'_{w'_q, x'_q} \quad \text{and} \quad A'_{w'_q, x'_q}(b^+\Delta) \supset \gamma'_{w'_q, x'_q} \cap \Delta_n(0, 5r/4),$$

for all $|w'_q|, |x'_q| < \varepsilon$. Consequently, the composition with h^{-1} yields a family of analytic discs $A_{w_q, x_q} := h^{-1} \circ A'_{w'_q, x'_q}$ which are half-attached to $\Sigma_{\gamma_{w_q, x_q}}$ and which satisfy similar properties, namely:

- (1) *The map $(w_q, x_q, \zeta) \mapsto A_{w_q, x_q}(\zeta)$ is \mathcal{C}^∞ -smooth and it provides a uniform family of \mathcal{C}^∞ embeddings of Δ into \mathbb{C}^n .*
 (2) *We have $A_{w_q, x_q}(1) = \gamma_{w_q, x_q}(0)$, and also:*

$$(5.9) \quad A_{w_q, x_q}(b^+\Delta) \subset \gamma_{w_q, x_q} \quad \text{and} \quad A_{w_q, x_q}(b^+\Delta) \supset \gamma_{w_q, x_q} \cap \Delta_n(0, r).$$

- (3) *$A_{w_q, x_q}(b^-\Delta) \subset\subset D \cup M_1^-$.*

This family A_{w_q, x_q} will be our starting point to study the envelope of holomorphy of (a certain subdomain of) the union of D together with a neighborhood Ω of M_1^- and an arbitrarily thin neighborhood of Σ_{w_q, x_q} . At first, we must include A_{w_q, x_q} into a larger family of discs obtained by sliding the half-attached part inside $\Sigma_{\gamma_{w_q, x_q}}$ along its complex tangential directions.

5.10. Deformation of half-attached analytic discs. To this aim, we introduce the equation $y = H_{w_q, x_q}(w, x)$ of $\Sigma_{\gamma_{w_q, x_q}}$, where the map $(w_q, x_q, w, x) \mapsto H_{w_q, x_q}(w, x)$ is of course \mathcal{C}^∞ and $\|H_{w_q, x_q} - H_{0,0}\|_{C^\infty(w, x)}$ is very small. Further, we need some formal notation. We denote $A_{w_q, x_q}(\zeta) := (w_{w_q, x_q}(\zeta), z_{w_q, x_q}(\zeta))$ and $A_{w_q, x_q}(1) [= \gamma_{w_q, x_q}(0)] := (w_{w_q, x_q}^1, z_{w_q, x_q}^1)$. For our discs, we shall choose the regularity $\mathcal{C}^{1, \alpha}$, $0 < \alpha < 1$, which is sufficient for our purposes. Let T_1 denote the Hilbert transform vanishing at 1 (see [Tu], [MP], [BER2]; by definition, T_1 is the unique (bounded, by a classical result) endomorphism of $\mathcal{C}^{1, \alpha}(\overline{\Delta}, \mathbb{R})$, $0 < \alpha < 1$, to itself such that $\phi + iT_1(\phi)$ extends holomorphically to Δ and $T_1\phi$ vanishes at $1 \in b\Delta$, i.e. $(T_1(\phi))(1) = 0$). Let φ^- and φ^+ be two \mathcal{C}^∞ functions on $b\Delta$ satisfying $\varphi^- \equiv 0$, $\varphi^+ \equiv 1$ on $b^+\Delta$ and $\varphi^- + \varphi^+ = 1$ on $b\Delta$. The fact that our discs are half attached to $\Sigma_{\gamma_{w_q, x_q}}$ can be expressed by saying that $y_{w_q, x_q}(\zeta) = \varphi^+(\zeta) H_{w_q, x_q}(w_{w_q, x_q}(\zeta), x_{w_q, x_q}(\zeta)) + \varphi^-(\zeta) y_{w_q, x_q}(\zeta)$ for all $\zeta \in b\Delta$ (cf. Aĭrapetyan [Aĭ]). Since the two functions x_{w_q, x_q} and y_{w_q, x_q} on $b\Delta$ are harmonic conjugates, the following (Bishop) equation is satisfied on $b\Delta$ by x_{w_q, x_q} :

$$(5.11) \quad x_{w_q, x_q}(\zeta) = -[T_1(\varphi^+ H_{w_q, x_q}(w_{w_q, x_q}, x_{w_q, x_q}))](\zeta) + \psi_{w_q, x_q}(\zeta) + x_{w_q, x_q}^1,$$

where we have set $\psi_{w_q, x_q}(\zeta) := -[T_1(\varphi^- y_{w_q, x_q})](\zeta)$. We want to perturb these discs A_{w_q, x_q} by “translating” them along the complex tangential directions to $\Sigma_{\gamma_{w_q, x_q}}$. Introducing a new parameter $\sigma \in \mathbb{C}^{n-1}$ with $|\sigma| < \varepsilon$, we can now include the discs A_{w_q, x_q} into a larger parameterized family $A_{w_q, x_q, \sigma}$ by solving the following perturbed Bishop equation on $b\Delta$ with parameters (w_q, x_q, σ) :

$$(5.12) \quad x_{w_q, x_q, \sigma}(\zeta) = -[T_1(\varphi^+ H_{w_q, x_q}(w_{w_q, x_q} + \sigma, x_{w_q, x_q, \sigma}))](\zeta) + \psi_{w_q, x_q}(\zeta) + x_{w_q, x_q}^1.$$

For instance, the existence and the $\mathcal{C}^{1, \beta}$ -smoothness (with $0 < \beta < \alpha$ arbitrary) of a solution $x_{w_q, x_q, \sigma}$ to (5.12) follows from Tumanov’s work [Tu3]. Clearly the solution disc $A_{w_q, x_q, \sigma}$ is half attached to $\Sigma_{\gamma_{w_q, x_q}}$. Differentiating the Bishop equation (5.12) with respect to σ , one sees that the derivatives $(\partial/\partial\sigma)(z_{w_q, x_q, \sigma})$ is uniformly small (cf. a similar computation in [Tu1, 2, 3]). In summary:

Lemma 5.13. *After shrinking perhaps ε , there exists a $\mathcal{C}^{1, \beta}$ -smooth mapping defined for $|w_q|, |x_q| < \varepsilon$ and for $\sigma \in \mathbb{C}^{n-1}$, $|\sigma| < \varepsilon$, $(w_q, x_q, \sigma, \zeta) \mapsto A_{w_q, x_q, \sigma}(\zeta)$, which is holomorphic with respect to ζ , and which fulfills the following conditions:*

- (1) $A_{w_q, x_q, 0} \equiv A_{w_q, x_q}$.
- (2) $A_{w_q, x_q, \sigma}(b^+\Delta) \subset \Sigma_{\gamma_{w_q, x_q}}$ for all σ .
- (3) The map $\mathbb{C}^{n-1} \times b^+\Delta \ni (\sigma, \zeta) \mapsto A_{0,0,\sigma}(\zeta) \in \Sigma_{\gamma_{0,0}}$ is a local $\mathcal{C}^{1, \beta}$ diffeomorphism from a neighborhood of 0×1 onto a neighborhood of $A_{0,0}(1) = 0$.

5.14. Preliminary to the continuity principle. We are now in position to state and to prove the main assertion of this paragraph. At first, we shall let the parameters (w_q, x_q, σ) range in certain new precise subdomains. We choose a positive $\delta < \varepsilon$ with the property that the range of the map in (3) above, when

restricted to $\{|\sigma| < \delta\} \times b^+\Delta$, contains the intersection of $\Sigma_{\gamma_{0,0}}$ with a small polydisc $\Delta_n(0, 2\eta)$, for some $\eta > 0$. Of course, there exists a constant $c > 1$, depending only on the Jacobian of the map in (3) at 0×1 such that $\frac{1}{c}\delta \leq \eta \leq c\delta$. Furthermore, since the boundary of the disc $A_{0,0,0}$ is transversal to $T_0^\varepsilon \Sigma_{\gamma_{0,0}}$ (whence $\frac{d}{d\lambda} A_{0,0,0}(\lambda)|_{\lambda \in \mathbb{R}, \lambda=1} \notin T_0 M$), then after shrinking a bit η if necessary, we can assume that the set $\{A_{0,0,\sigma}(\zeta) : |\sigma| < \delta, \zeta \in \Delta \cap \Delta(1, \delta)\}$ contains and foliates by half analytic discs the whole lower side $\Delta_n(0, 2\eta) \cap \Sigma_{\gamma_{0,0}}^-$.

Remark. Of course, the side $\Sigma_{\gamma_{0,0}}^-$ is “the same side” as M^- , i.e. the side of $\Sigma_{\gamma_{0,0}}$ where the greatest portion of D lies, although D is in general *not* entirely contained in $\Sigma_{\gamma_{0,0}}^-$, because the Segre varieties $S_{\bar{q}}$ for $q \in \gamma_{0,0}$ may well intersect D , as is known.

As in §2, we now fix a neighborhood Ω of M_1^- in \mathbb{C}^n to which all the components of the reflection function extend holomorphically as in Lemma 4.5. We have already shown that the half parts $A_{w_q, x_q, \sigma}(b^+\Delta)$ are all contained in $\Sigma_{\gamma_{w_q, x_q}}$ (hence in arbitrarily thin neighborhoods of it). It remains now to control the half parts $A_{w_q, x_q, \sigma}(b^-\Delta)$. Using property (3) after (5.9) for $(w_q, x_q) = (0, 0)$, namely $A_{0,0,0}(b^-\Delta) \subset \subset D \cup M_1^-$, it is clear that, after shrinking δ if necessary, then we can insure that $A_{0,0,\sigma}(b^-\Delta) \subset \subset D \cup \Omega$ for all $\sigma \in \mathbb{C}^{n-1}$ with $|\sigma| < \delta$. Of course, this shrinking will result in a simultaneous shrinking of η , and we still have the important supclusion: $\{A_{0,0,\sigma}(\zeta) : |\sigma| < \delta, \zeta \in \Delta \cap \Delta(1, \delta)\} \supset \Delta_n(0, 2\eta) \cap \Sigma_{\gamma_{0,0}}^-$. Finally, shrinking again ε if necessary, we then come to a situation that we may summarize:

(1) For all $|w_q|, |x_q| < \varepsilon$, we have:

$$(5.15) \quad \{A_{w_q, x_q, \sigma}(\zeta) : |\sigma| < \delta, \zeta \in \Delta \cap \Delta(1, \delta)\} \supset \Delta_n(0, \eta) \cap \Sigma_{\gamma_{w_q, x_q}}^-.$$

(2) $A_{w_q, x_q, \sigma}(b^+\Delta) \subset \Sigma_{\gamma_{w_q, x_q}}$ and $A_{w_q, x_q, \sigma}(b^-\Delta) \subset \subset D \cup \Omega$, for all $|\sigma| < \delta$.

Remark. Shrinking for the last time δ and ε if necessary, we can further insure that all the discs $A_{w_q, x_q, \sigma}$ are embeddings of $\overline{\Delta}$ in \mathbb{C}^n , which will be convenient to apply the continuity principle. If ε is small enough, we can also insure that the intersection of D with $\Delta_n(0, \eta) \cap \Sigma_{\gamma_{w_q, x_q}}^-$ is connected for all $|w_q|, |x_q| < \varepsilon$.

5.16. Envelopes of holomorphy. We are now in position to state and to prove the main assertion of this paragraph. Especially, the following lemma will be applied to each member of the collection $\{\Theta'_\beta(h(t))\}_{\beta \in \mathbb{N}^{n-1}}$.

Lemma 5.17. *Let $\delta, \eta, \varepsilon > 0$ be as above. If a holomorphic function $\psi \in \mathcal{O}(D)$ extends holomorphically to a neighborhood $\omega(\Sigma_{\gamma_{w_q, x_q}})$, then there exists a unique holomorphic function $\Psi \in \mathcal{O}(D \cup [\Delta_n(0, \eta) \cap \Sigma_{\gamma_{w_q, x_q}}^-])$ such that $\Psi|_D \equiv \psi$.*

Proof. Here, we fix w_q and x_q . First, we notice that the assumption entails that ψ extends to a holomorphic function in domains of the form $D_1 \cup \omega(\Sigma_{\gamma_{w_q, x_q}})$, for any subdomain $D_1 \subset D$ is such that $D_1 \cap \omega(\Sigma_{\gamma_{w_q, x_q}})$ is connected. Clearly, D_1 can be chosen so that $D_1 \cup \omega(\Sigma_{\gamma_{w_q, x_q}})$ contains the union of the boundaries $A_{w_q, x_q, \sigma}(b\Delta)$, for $|\sigma| < \delta$. Next, we notice that for various σ 's, all our discs are clearly *analytically isotopic* to $A_{w_q, x_q, 0}$ (see [Me2]). Since this last disc is clearly isotopic to a point in D_1 (just do the isotopy by shrinking A_{w_q, x_q} to its center point $A_{w_q, x_q}(0)$), we can apply the continuity principle in the version given by Lemma 3.2 in [Me2] to deduce that, for all σ , there exists a holomorphic function in a neighborhood of

$A_{w_q, x_q, \sigma}(\bar{\Delta})$ which coincides with ψ in a neighborhood of $A_{w_q, x_q, \sigma}(b\Delta)$. Then some arguments similar to those given in [Me2, p. 40] apply to deduce that there exists a unique holomorphic function $\chi \in \mathcal{O}(\Delta_n(0, \eta) \cap \Sigma_{\gamma_{w_q, x_q}}^-)$ which coincides with ψ in a neighborhood of γ_{w_q, x_q} (the fact that the map $\Delta_{n-1}(0, \delta) \times \Delta \cap \Delta(1, \delta) \ni (\sigma, \zeta) \mapsto A_{w_q, x_q, \sigma}(\zeta) \in \mathbb{C}^n$ is an embedding is an important property which insures uniqueness of the extension). Finally, using the connectedness of $\Delta_n(0, \eta) \cap \Sigma_{\gamma_{w_q, x_q}}^- \cap D$ and the principle of analytic continuation, we get the holomorphic function $\Psi \in \mathcal{O}(D \cup [\Delta_n(0, \eta) \cap \Sigma_{\gamma_{w_q, x_q}}^-])$, which completes the proof of Lemma 5.17. \square

§6. HOLOMORPHIC EXTENSION TO A LEVI-FLAT UNION OF SEGRE VARIETIES

6.1. Straigtenings. For each parameter $(w'_{q'}, x'_{q'})$, we have considered the analytic arc $\gamma'_{w'_{q'}, x'_{q'}}$ defined by (5.2). To this family of analytic arcs we can clearly associate a family of normalizing coordinates as follows.

Lemma 6.2. *If $\varepsilon \ll r$ is small enough, there exists a \mathbb{C}^ω -parameterized family of biholomorphic mappings $\Phi'_{w'_{q'}, x'_{q'}}$ of $\Delta_n(0, 2r)$ which straightens $\gamma'_{w'_{q'}, x'_{q'}}$ to the x' -axis, such that the image $M'_{w'_{q'}, x'_{q'}} := \Phi'_{w'_{q'}, x'_{q'}}(M')$ is a closed \mathbb{C}^ω hypersurface of $\Delta_n(0, r)$ close to $M'_{0,0}$ in \mathbb{C}^ω norm which is given by an equation of the form $\bar{z}' = \Theta'_{w'_{q'}, x'_{q'}}(\bar{w}', t')$, with $\Theta'_{w'_{q'}, x'_{q'}}(\bar{w}', t')$ converging normally in the polydisc $\Delta_{2n-1}(0, r)$ and satisfying $\Theta'_{w'_{q'}, x'_{q'}}(0, t') \equiv 0$.*

6.3. Different reflection functions. Let us developpe this equation in the form :

$$(6.4) \quad \bar{z}' = \Theta'_{w'_{q'}, x'_{q'}}(\bar{w}', t') = \sum_{\beta \in \mathbb{N}^{n-1}} \bar{w}'^\beta \Theta'_{w'_{q'}, x'_{q'}, \beta}(t').$$

We denote by $h_{w'_{q'}, x'_{q'}} = (g_{w'_{q'}, x'_{q'}}, f_{w'_{q'}, x'_{q'}})$ the map in these coordinate systems. To all such coordinates are therefore associated *different reflection functions* by :

$$(6.5) \quad \mathcal{R}'_{w'_{q'}, x'_{q'}, h_{w'_{q'}, x'_{q'}}}(t, \bar{v}') := \bar{\mu}' - \sum_{\beta \in \mathbb{N}^{n-1}} \bar{\lambda}'^\beta \Theta'_{w'_{q'}, x'_{q'}, \beta}(h_{w'_{q'}, x'_{q'}}(t)).$$

6.6. Holomorphic extension to a Levi-flat hat. We now come to the main construction of this paragraph. Let $E'_{M'} = \mathcal{E}' \cap M' \subset M'$ be the real analytic subset of §3. Then we claim that $E'_{M'}$ is of real dimension $\leq 2n - 3$. Indeed, the complex dimension of \mathcal{E}' is $\leq n - 1$. If $E'_{M'}$ would contain a $(2n - 2)$ -dimensional real analytic piece of manifold, this piece would certainly be a complex analytic hypersurface contained in M' , contradicting the fact that M' is minimal at every point. Because the codimension of $E'_{M'}$ in M' is ≥ 2 , then for almost all $(w'_{q'}, x'_{q'})$ (i.e. except a closed set of zero Lebesgue measure in the parameter space), then the intersection $\gamma'_{w'_{q'}, x'_{q'}} \cap E'_{M'}$ is empty. In this situation, we have :

Lemma 6.7. *For every $(w'_{q'}, x'_{q'})$ with $\gamma'_{w'_{q'}, x'_{q'}} \cap E'_{M'} = \emptyset$ and $\gamma'_{w'_{q'}, x'_{q'}} \subset M_1'^-$, then all the components $\Theta'_{w'_{q'}, x'_{q'}, \beta}(h_{w'_{q'}, x'_{q'}}(t))$ of $\mathcal{R}'_{w'_{q'}, x'_{q'}, h_{w'_{q'}, x'_{q'}}}(t, \bar{v}')$ extend to be holomorphic in a neighborhood $\omega(\Sigma_{\gamma_{w_q, x_q}})$ of $\Sigma_{\gamma_{w_q, x_q}}$ in \mathbb{C}^n .*

Remark. To prove Lemma 6.7, we need the existence of a map H as in Lemma 3.7 at every point of γ_{w_q, x_q} . This is why we require $\gamma_{w_q, x_q} \cap E'_{M'} \neq \emptyset$. However, as h

is \mathcal{C}^∞ in Theorem 1.8, the condition $\gamma'_{w'_q, x'_q} \cap E'_{M'} = \emptyset$ can be removed, thanks to the remark after Lemma 3.7. But we have in mind the same theorem with h being only of class $\mathcal{C}^{1,\alpha}$ (see Theorem 7.12 below). Thus, we will conduct the proof taking into account $E'_{M'}$, even when h is \mathcal{C}^∞ .

Proof. After a biholomorphic change of coordinates near M , we can assume from the beginning that $dh(0) = \text{Id}$ and that $\Theta(\bar{w}, t)$ converges normally in $\Delta_{2n-1}(0, 4r)$. At first, we shall establish our main crucial observation as follows.

Lemma 6.8. *If $|w_q|, |x_q| < \varepsilon$ and ε is sufficiently small, then all the components $\Theta'_\beta(h(t))$ extend as CR functions of class \mathcal{C}^∞ over $\Sigma_{\gamma_{w_q, x_q}} \cap \Delta_n(0, r)$.*

Now we remind the reader that the $\Theta'_\beta(h(t))$'s already extend holomorphically to a neighborhood $\omega(\gamma_{w_q, x_q}) \subset \Omega$ of $\gamma_{w_q, x_q} \subset M_1^-$ in \mathbb{C}^n , by construction. Taking Lemma 6.8 for granted we shall then complete the proof of Lemma 6.7 by an application of the following statement (a minor generalization of the Hanges-Treves extension theorem to parametrized hypersurfaces which also holds in $\mathcal{C}^{1,\alpha}$): \square

Lemma 6.9. *Let Σ be a \mathcal{C}^∞ -smooth Levi-flat hypersurface in \mathbb{C}^n ($n \geq 2$). If a continuous CR function ψ extends holomorphically to a neighborhood \mathcal{U}_p of a point p belonging to a leaf \mathcal{F}_Σ of Σ , then ψ extends holomorphically to a neighborhood $\omega(\mathcal{F}_\Sigma)$ of \mathcal{F}_Σ in \mathbb{C}^n . The size of this neighborhood $\omega(\mathcal{F}_\Sigma)$ depends on the size of \mathcal{U}_p and is stable under sufficiently small (even non-Levi-flat) perturbations of Σ .*

Proof of Lemma 6.8. Let $\bar{L}_1, \dots, \bar{L}_{n-1}$ be the commuting basis of $T^{0,1}M$ given by $\bar{L}_j = \frac{\partial}{\partial \bar{w}_j} + \Theta_{\bar{w}_j}(\bar{w}, t) \frac{\partial}{\partial \bar{z}}$, $1 \leq j \leq n-1$. Clearly, the coefficients of these vectors fields converge normally in the polydisc $\Delta_{2n-1}(0, 4r)$. By the diffeomorphism assumption, we have $\det(\bar{L}_j \bar{g}_k)_{1 \leq j, k \leq n-1}(0) \neq 0$. We shall denote this determinant by:

$$(6.10) \quad \underline{\mathcal{D}}(\bar{w}, t, \{\partial_{\bar{L}_j} \bar{g}_k(\bar{t})\}_{1 \leq j \leq n, 1 \leq k \leq n-1}).$$

Here, t belongs to M and the function $\underline{\mathcal{D}}$ is holomorphic in its variables. Replacing z by $\bar{\Theta}(w, \bar{t})$ in $\underline{\mathcal{D}}$, we can write $\underline{\mathcal{D}}$ in the form $\mathcal{D}(w, \bar{t}, \{\partial_{\bar{L}_j} \bar{g}_k(\bar{t})\}_{1 \leq j \leq n, 1 \leq k \leq n-1})$, where \mathcal{D} is holomorphic in its variables. Shrinking $r > 0$ if necessary, we may assume that for all fixed coordinate point $\bar{t}_q = (\bar{w}_q, \bar{z}_q) \in M$ with $|\bar{t}_q| < r$, then:

- (1) *The polarization $\mathcal{D}(w, \bar{t}_q, \{\partial_{\bar{L}_j} \bar{g}_k(\bar{t}_q)\}_{1 \leq j \leq n, 1 \leq k \leq n-1})$ is convergent on the Segre variety $S_{\bar{t}_q} \cap \Delta_n(0, 2r) = \{(w, z) \in \Delta_n(0, 2r) : z = \bar{\Theta}(w, \bar{t}_q)\}$, i.e. is convergent with respect to w for $|w| < 2r$.*
- (2) *This expression $\mathcal{D}(w, \bar{t}_q, \{\partial_{\bar{L}_j} \bar{g}_k(\bar{t}_q)\}_{1 \leq j \leq n, 1 \leq k \leq n-1})$ does not vanish at any point of $S_{\bar{t}_q} \cap \Delta_n(0, 2r)$, i.e. does not vanish for all $|w| < 2r$.*

Let us choose (w'_q, x'_q) satisfying $\gamma'_{w'_q, x'_q} \subset M_1'^-$, with $|w'_q|, |x'_q| < \varepsilon$. We pick the corresponding parameter (w_q, x_q) with $|w_q|, |x_q| < \varepsilon$. By the choice of $\Phi'_{w'_q, x'_q}$, we then have $g_{w'_q, x'_q}(\gamma_{w_q, x_q}(s)) = 0$ for all $s \in \mathbb{R}$ with $|s| \leq 2r$. This property will be crucial. As h is only of class \mathcal{C}^∞ over M , to apply the tangential Cauchy-Riemann derivations and to make a polarization afterwards, we need at first replace $h_{w'_q, x'_q}$ in a neighborhood of $\gamma_{w_q, x_q}(s)$ by a local *holomorphic* equivalence $H_{w'_q, x'_q} : (M, \gamma_{w_q, x_q}(s)) \rightarrow (M', \gamma'_{w'_q, x'_q}(s'))$ given by Lemma 3.7 (with $k = 1$) satisfying:

- (1) $H_{w'_q, x'_q, s}(\gamma_{w_q, x_q}(s)) = h_{w'_q, x'_q}(\gamma_{w_q, x_q}(s)) = (0, f_{w'_q, x'_q}(\gamma_{w_q, x_q}(s)))$.

- (2) $dH_{w_{q'}, x_{q'}}(\gamma_{w_q, x_q}(s)) = dh_{w_{q'}, x_{q'}}(\gamma_{w_q, x_q}(s)).$
 (3) $\Theta'_{w_{q'}, x_{q'}, \beta}(H_{w_{q'}, x_{q'}}(t)) \equiv \Theta'_{w_{q'}, x_{q'}, \beta}(h_{w_{q'}, x_{q'}}(t))$ for all t close to $\gamma_{w_q, x_q}(s)$.

For this application of Lemma 3.7, we fix the parameter s , namely we fix the point $\gamma_{w_q, x_q}(s)$ (but according to Lemma 3.7, when s varies, for all $k \in \mathbb{N}$, there will exist such holomorphic equivalence $H_{w_{q'}, x_{q'}, s}$ depending \mathcal{C}^k -smoothly with respect to s , a property which we shall need below). If we write $H_{w_{q'}, x_{q'}} := (G_{w_{q'}, x_{q'}}, F_{w_{q'}, x_{q'}})$, then applying the tangential Cauchy-Riemann operators $\bar{L}_1^{\beta_1} \dots \bar{L}_{n-1}^{\beta_{n-1}}$, $\beta \in \mathbb{N}^{n-1}$ to the identity :

$$(6.11) \quad \bar{F}_{w_{q'}, x_{q'}}(\bar{t}) = \Theta'_{w_{q'}, x_{q'}, \beta}(\bar{G}_{w_{q'}, x_{q'}}(\bar{t}), H_{w_{q'}, x_{q'}}(t)),$$

which holds for t in a neighborhood of $\gamma_{w_q, x_q}(s)$, we get by a classical calculation (see [BJT], [BR1,4], [BER2], [Me4]) an infinite family of identities of the following kind, for all $\beta \in \mathbb{N}^{n-1}$ (the case $\beta = 0$ simply means (6.11)) :

$$(6.12) \quad \left\{ \begin{aligned} & \Theta'_{w_{q'}, x_{q'}, \beta}(H(t)) + \sum_{\kappa \in \mathbb{N}_*^{n-1}} \frac{(\beta + \kappa)!}{\beta! \kappa!} \bar{G}(\bar{t})^\gamma \Theta'_{w_{q'}, x_{q'}, \beta + \kappa}(H(t)) = \\ & = \frac{\mathcal{T}_\beta(w, \bar{t}, \{\partial_{\bar{t}_j}^k \bar{H}_j(\bar{t})\}_{1 \leq j \leq n, |\kappa| \leq |\beta|})}{[\mathcal{D}(w, \bar{t}, \{\partial_{\bar{t}_j} \bar{G}_k(\bar{t})\}_{1 \leq j \leq n, 1 \leq k \leq n-1})]^{2|\beta|-1}}. \end{aligned} \right.$$

Here the \mathcal{T}_β 's are holomorphic with respect to (w, \bar{t}) and relatively polynomial with respect to the jets $\{\partial_{\bar{t}_j} \bar{G}_k(\bar{t})\}_{1 \leq j \leq n, 1 \leq k \leq n-1}$. Also, the variable t runs in M in a neighborhood of $\gamma_{w_q, x_q}(s)$. To lighten a bit the notation, we have dropped the subscript of $H_{w_{q'}, x_{q'}}$ exceptionally (however, we will anyway keep this subscript in the sequel). Now, we remind the reader that all the functions $t \mapsto \Theta'_{w_{q'}, x_{q'}, \beta}(h'_{w_{q'}, x_{q'}, \beta}(t))$ are already real analytic in a neighborhood of γ_{w_q, x_q} , since $\gamma_{w_q, x_q} \subset M_1^-$. Let us denote them by :

$$(6.13) \quad \varphi'_{w_{q'}, x_{q'}, \beta}(t) := \Theta'_{w_{q'}, x_{q'}, \beta}(h_{w_{q'}, x_{q'}}(t)) \equiv \Theta'_{w_{q'}, x_{q'}, \beta}(H_{w_{q'}, x_{q'}}(t)).$$

We can replace them directly in the left hand side of (6.12). Written in this form, (6.12) then involves only functions which are holomorphic in t and in \bar{t} , for (t, \bar{t}) running in a neighborhood of $(\gamma_{w_q, x_q}(s), \overline{\gamma_{w_q, x_q}(s)})$. Thus, we can complexify (6.12), by replacing (t, \bar{t}) by $(t, \tau) \in \mathcal{M}$ close to $(\gamma_{w_q, x_q}(s), \overline{\gamma_{w_q, x_q}(s)})$, where \mathcal{M} is the extrinsic complexification of M , given in coordinates $(t, \tau) = (w, z, \zeta, \xi)$ by the holomorphic equation $\xi = \Theta(\zeta, t)$. Choosing (t, τ) of the form $(w, z, \overline{\gamma_{w_q, x_q}(s)})$, namely, choosing (w, z) to belong to the Segre variety $S_{\overline{\gamma_{w_q, x_q}(s)}}$, and using the important fact that $\bar{g}(\overline{\gamma_{w_q, x_q}(s)}) = \bar{G}(\overline{\gamma_{w_q, x_q}(s)}) = 0$ for such a fixed s (this is a crucial point, since it entails that the queue sum $\sum_{\kappa \in \mathbb{N}_*^{n-1}}$ in (6.12) disappears), we obtain, for $t \in S_{\overline{\gamma_{w_q, x_q}(s)}}$ close to $\gamma_{w_q, x_q}(s)$, i.e. for $t = (w, \bar{\Theta}(w, \overline{\gamma_{w_q, x_q}(s)}))$:

$$(6.14) \quad \varphi'_{w_{q'}, x_{q'}, \beta}(w, \bar{\Theta}(w, \overline{\gamma_{w_q, x_q}(s)})) = \frac{\mathcal{T}_\beta(w, \bar{t}, \{\partial_{\bar{t}_j}^\alpha \bar{H}_k(\bar{t})\}_{1 \leq k \leq n, |\alpha| \leq |\beta|})}{[\mathcal{D}(w, \bar{t}, \{\partial_{\bar{t}_j} \bar{g}_k(\bar{t})\}_{1 \leq j \leq n, 1 \leq k \leq n-1})]^{2|\beta|-1}}.$$

Here, we have directly replaced $\partial_{\bar{t}_j} \bar{G}_k(\bar{t})$ by $\partial_{\bar{t}_j} \bar{g}_k(\bar{t})$ in the denominator, which is allowed, since the one-jets of $\bar{H}_{w_{q'}, x_{q'}}$ and of $\bar{h}_{w_{q'}, x_{q'}}$ coincide at $\overline{\gamma_{w_q, x_q}(s)}$. Thanks

to (1) and (2) after (6.10), we see that the right hand side of (6.14) converges with respect to w for all $|w| < r$. We thus have got that the Taylor series of the components of the reflection function converge on each leaf of $\Sigma_{\gamma_{w_q, x_q}} \cap \Delta_n(0, r)$. It remains finally to verify that the right hand sides of (6.14) depend in fact in a C^∞ fashion with respect to s . Applying Lemma 3.7, we see that we can construct convergent equivalences $H_{w'_{q'}, x'_{q'}, s}$ depending C^k -smoothly with respect to s . Replacing it in (6.14), we get the desired transversal smooth dependence over $\Sigma_{\gamma_{w_q, x_q}}$. This completes the proofs of Lemmas 6.7 and 6.8. \square

Remark. As h is C^∞ in our Theorems 1.8 and 1.13, it is in fact superfluous to replace $h_{w'_{q'}, x'_{q'}}$ by a holomorphic local equivalence $H_{w'_{q'}, x'_{q'}, s}$ at $\gamma_{w_q, x_q}(s)$, because infinite derivations are allowed and we see in this case that (6.12) has a sense with h replacing H . However, we have conducted the above proof so, because we have in mind to generalize our results for a $C^{1, \alpha}$ mapping, see §7.11 below.

§7. RELATIVE POSITION OF THE NEIGHBOURING SEGRE VARIETIES

7.1. Intersection of Segre varieties. We are now in position to complete the proof of Theorem 1.8. Using Lemma 5.17 and Lemma 6.7, it remains to show that our functions $\Theta'_{w'_{q'}, x'_{q'}, \beta}$ extend holomorphically at 0, for γ_{w_q, x_q} chosen conveniently.

For this choice, we are led to the following dichotomy: either $S_{\bar{0}} \cap M_1^- = \emptyset$ in a sufficiently small neighborhood of 0 or there exists a sequence $(q_k)_{k \in \mathbb{N}}$ of points of $S_{\bar{0}} \cap M_1^-$ tending towards 0. In the first case, we shall distinguish two sub-cases. Either $S_{\bar{0}}$ lies under M_1^- or it lies above M_1^- . Let us write this more precisely. We can choose a C^∞ -smooth hypersurface H_1 transverse to M at 0 with H_1 satisfying $H_1 \cap M = M_1$ and $H_1^- \cap M = M_1^-$. Thus H_1 together with M divides \mathbb{C}^n near 0 in four connected parts. We wanted to say that either $S_{\bar{0}} \cap H_1^-$ is contained in the lower left quadrant $H_1^- \cap M^-$ or it is contained in the upper left quadrant $H_1^- \cap M^+$. To summarize, we have distinguished three cases: *Case I*: $S_{\bar{0}} \cap M_1^- \neq \emptyset$ in every neighborhood of 0; *Case II*: $S_{\bar{0}} \cap H_1^- \subset M^-$; *Case III*: $S_{\bar{0}} \cap H_1^- \subset M^+$. In the first two cases, the Segre variety $S_{\overline{\gamma_{w_q, x_q}(0)}}$ will intersect $D \cup \Omega$ and the neighborhood $\omega(\Sigma_{\gamma_{w_q, x_q}})$ will always contain 0: the extension will then be an easy direct application of the Hanges-Treves extension theorem (see the details below). The third case could be a priori the most delicate one. But we can already delineate the following crucial geometric property, which says that Lemma 5.17 will apply.

Lemma 7.2. *If $S_{\bar{0}} \cap H_1^-$ is contained in M^+ , then 0 lies in the lower side $\Sigma_{\gamma_{w_q, x_q}}^-$ for every arc $\gamma_{w_q, x_q} \subset \subset M_1^-$.*

Proof. The real equation of M is given by $y = h(w, \bar{w}, x)$, where h is a certain converging real power series satisfying $h(0) = 0$, $dh(0) = 0$ and $h(w, 0, x) \equiv 0$. We can assume that the “minus” side $D = M^-$ of automatic extension of CR functions is given by $\{y < h(w, \bar{w}, x)\}$. Replacing x by $(z + \bar{z})/2$ and y by $(z - \bar{z})/2i$, and solving with respect to z , we get for M an equation as above, say $z = \bar{z} + i\Xi(w, \bar{t})$ (we have $\Theta(w, \bar{t}) \equiv \bar{z} + i\Xi(w, \bar{t})$ in our previous notations), with $\Xi(0, \bar{t}) \equiv 0$. Clearly, every such arc γ_{w_q, x_q} contains a point $p \in M$ of coordinates $(w_p, 0 + ih(w_p, \bar{w}_p, 0))$ with $u_{1,p} < 0$ (indeed, by construction, these arcs are all elongated and almost directed along the x -coordinate lines since $dh(0) = \text{Id}$). By assumption, we have $h(w_p, \bar{w}_p, 0) < 0$. Then the Segre variety $S_{\bar{p}}$ (which is a leaf of $\Sigma_{\gamma_{w_q, x_q}}$), has the equation $z = -ih(w_p, \bar{w}_p, 0) + i\Xi(w, \bar{w}_p, -ih(w_p, \bar{w}_p, 0))$. Therefore, the intersection

point $\{w = 0\} \cap S_{\bar{p}} \subset \Sigma_{\gamma_{w_q, x_q}}$ whose coordinates are $(0, -ih(w_p, \bar{w}_p, 0))$ clearly lies over the origin, which completes the proof of Lemma 7.2. \square

7.3. Extension across $(M, 0)$ of the components $\Theta'_{w'_{q'}, x'_{q'}, \beta}$. We are now prepared to complete the proof of Theorem 1.8. We first choose $\delta, \eta, \varepsilon$ and various $|w_q|, |x_q| < \varepsilon$ as in Lemma 5.17 and we consider the associated arcs γ_{w_q, x_q} , $\gamma'_{w'_{q'}, x'_{q'}}$, the associated mapping $h_{w'_{q'}, x'_{q'}}$ and the associated reflection function $\mathcal{R}'_{w'_{q'}, x'_{q'}, h_{w'_{q'}, x'_{q'}}}$. By Lemmas 6.7 and 5.17, for each such choice of (w_q, x_q) , then all the components $\Theta'_{w'_{q'}, x'_{q'}, \beta}$ extend holomorphically to $D \cup [\Sigma_{\gamma_{w_q, x_q}}^- \cap \Delta_n(0, \eta)]$. Our goal is to show that for suitably chosen γ_{w_q, x_q} in Cases I, II and III, then the components $\Theta'_{w'_{q'}, x'_{q'}, \beta}$ extend holomorphically across $(M, 0)$ (afterwards, in Lemma §7.10 below, we shall establish the desired final Cauchy estimate).

7.4. Case I. In Case I, we choose an arc γ_{w_q, x_q} passing through one of the points $q_k \in M \cap S_0$ sufficiently close to 0, with $|q_k| < \eta/2$ and with $|w_q|, |x_q| < \varepsilon$. Since $q_k \in M_1^- \cap S_0$, we have $0 \in S_{\bar{q}_k}$. However, it certainly can happen that q_k belongs to $E'_{M'}$ and then Lemma 6.7 fails to apply. Fortunately, we can shift slightly γ_{w_q, x_q} in order that $\gamma'_{w'_{q'}, x'_{q'}} \cap E'_{M'} = \emptyset$ (remember: $h(\gamma_{w_q, x_q}) = \gamma'_{w'_{q'}, x'_{q'}}$). Since the Hausdorff dimension of $E'_{M'}$ does not exceed $2n - 3$, it is clear that we can choose such an arc γ_{w_q, x_q} with q_k arbitrarily to close γ_{w_q, x_q} and $\gamma'_{w'_{q'}, x'_{q'}} \cap E'_{M'}$ empty. We let \tilde{q} denote a point of γ_{w_q, x_q} which is the nearest to q_k . We have $|q_k - \tilde{q}| \ll \varepsilon, \eta, \delta$. By Lemma 4.5, the components of the reflection function associated with our fixed coordinate system vanishing at 0 (*i.e.* for $(w'_{q'}, x'_{q'}) = (0, 0)$) all converge in the polydisc $\Delta_n(\tilde{q}, r(\tilde{q}))$ and they satisfy a Cauchy estimate there. This implies that the components of the reflection function associated with $\gamma'_{w'_{q'}, x'_{q'}}$ converge in the polydisc $\Delta_n(q_k, r(q_k)/2)$ and satisfy a Cauchy estimate there (remember: $q \mapsto r(q)$ is lower semi-continuous). To summarize, we have got:

Lemma 7.5. *There exist arcs γ_{w_q, x_q} passing through a point \tilde{q} arbitrarily close to q_k such that the associated components $\Theta'_{w'_{q'}, x'_{q'}, \beta}$ extend holomorphically to*

$$(7.6) \quad D \cup [\Sigma_{\gamma_{w_q, x_q}}^- \cap \Delta_n(0, \eta)] \cup \Delta_n(q_k, r(q_k)/2).$$

Applying then Lemma 6.9, we see that the neighborhood $\omega(S_{\tilde{q}})$ to which holomorphic extension holds will contain 0 if such a \tilde{q} is sufficiently close to q_k (remember: $0 \in S_{\bar{q}_k}$). Further, choosing this neighborhood $\omega(S_{\tilde{q}})$ to be of nice tubular form (shrinking it a bit if necessary), we can certainly assure that its intersection with the open set (7.6) is connected. Case I is done.

7.7. Case II. Case II, is treated almost the same way. Since $S_0 \cap H_1^-$ is contained in D , we can choose a fixed point \tilde{q} of S_0 which belongs to $\Delta_n(0, \eta/2)$. Of course, there exists a positive radius $\tilde{r} > 0$ such that the polydisc $\Delta_n(\tilde{q}, \tilde{r})$ is contained in D in which the components $\Theta'_{w'_{q'}, x'_{q'}, \beta}$ satisfy a Cauchy estimate, for all $|w'_{q'}|, |x'_{q'}|$ sufficiently small. We thus come down to a situation similar to that of Lemma 7.5. Case II is done.

7.8. Case III. For Case III, thanks to Lemma 7.2, we know already that 0 belongs to the lower side $\Sigma_{\gamma_{w_q, x_q}}^-$. Thus Case III follows immediately from the application of Lemmas 6.7 and 5.17 summarized in §7.3 above. Case III is done.

7.9. Extension across M of the reflection function. To deduce that the reflection function extends holomorphically at 0, it remains to establish a final Cauchy estimate.

Lemma 7.10. *Let $|w'_{q'}|, |x'_{q'}| < \varepsilon$ and assume that the components $\Theta'_{w'_{q'}, x'_{q'}, \beta}(h(t))$ extend as holomorphic functions $\varphi'_{w'_{q'}, x'_{q'}, \beta}(t)$ at 0. Then there exist constants $C_0 > 0$, $r_0 > 0$ such that $|\varphi'_{w'_{q'}, x'_{q'}, \beta}(t)| < C_0^{|\beta|+1}$ for $|t| < r_0$.*

Proof. The formal power series $h_{\mathcal{F}}(t)$ at 0 satisfies $\Theta'_{w'_{q'}, x'_{q'}, \beta}(h_{\mathcal{F}}(t)) \equiv \varphi'_{w'_{q'}, x'_{q'}, \beta}(t)$. Thanks to the Artin approximation theorem, there exists a converging power series $H(t)$ such that $\Theta'_{w'_{q'}, x'_{q'}, \beta}(H(t)) \equiv \varphi'_{w'_{q'}, x'_{q'}, \beta}(t)$. Then the easy Cauchy estimate for the composition of two holomorphic functions yields Lemma 7.10. The proof of Theorem 1.8 is complete. \square

7.11. Mappings of lesser regularity. Clearly, our constructions in §2,3,4,5,6,7 are valid for a $\mathcal{C}^{1,\alpha}$ ($0 < \alpha < 1$) CR diffeomorphism h , except the last step, namely except the proof of the Cauchy estimates for our functions $\varphi'_{w'_{q'}, x'_{q'}, \beta}(t)$, whose holomorphicity is established along the same lines (for the $\mathcal{C}^{1,\alpha}$ version of the Hanges-Treves theorem, one has to use the work of Tumanov [Tu2]). However, in the holomorphically nondegenerate case, we can complete the following:

Theorem 7.12. *If h is a $\mathcal{C}^{1,\alpha}$ -smooth CR-diffeomorphism, if (M, p) is minimal and if (M', p) is holomorphically nondegenerate, then the reflection function \mathcal{R}'_h extends holomorphically to a neighborhood of $p \times \overline{p'}$.*

(However, we are still unable to conclude that h is real analytic.) Indeed, first of all, it is not difficult to deduce then that such a $\mathcal{C}^{1,\alpha}$ map extends as a correspondence at the point $p_1 \in M_1$ (by taking the uniquely defined irreducible component of \mathcal{C}'_h which contains the graph of h). Consequently, we get polynomial identities with coefficients holomorphic in a neighborhood of p_1 for the components of the mapping h . Finally, to prove that the reflection function extends holomorphically at p_1 , namely to get the desired Cauchy estimates, we can then apply the same scheme of proof as for instance in [BJT] or [DP].

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